Iowa State University
Digital Repository

Retrospective Theses and Dissertations

Iowa State University Capstones, Theses and Dissertations

1983

Life analysis of distributions based on item value

Slamet
Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Industrial Engineering Commons

Recommended Citation
Slamet, "Life analysis of distributions based on item value " (1983). *Retrospective Theses and Dissertations*. 8961.
https://lib.dr.iastate.edu/rtd/8961

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.

المنارة للاستشارات

www.manaraa.com

INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.

2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.

3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of "sectioning" the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.

4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.

5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.

University Microfilms International

300 N. Zeeb Road
Ann Arbor, MI 48106

8407125

Slamet

LIFE ANALYSIS OF DISTRIBUTIONS BASED ON ITEM VALUE

Iowa State University PH.D. 1983

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

Life analysis of distributions

based on item value

by

Slamet

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Department: Industrial Engineering

Major: Engineering Valuation

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1983

TABLE OF CONTENTS

I. INTRODUCTION

Estimates of the mortality behavior of a property are useful for calculating depreciation and making management decisions relating to property.

Mortality characteristics usually have been ascertained through the use of one of three approaches: the actuarial methods, the semi-actuarial methods, and the forecast method. The actuarial methods are distinguished from the other two groups in that they require a knowledge of the properties age at retirement. Analysis by the actuarial methods yield a life dispersion pattern and, accordingly, an estimate of average service life.

The actuarial procedures can be applied only to plant accounts that have complete age identification. This limitation encouraged the development of the simulated-plant-record or SPR methods which are semi-actuarial methods.

The semi-actuarial methods only require a knowledge of annual retirements, annual balances, and annual additions. The SPR methods are simply trial and error procedures in which an attempt is made to simulate some portion of a plant accounting record that may or may not permit age identification of plant retirements.

The forecast method differs from the other two in that it does not require numerical data prior to estimating a life dispersion pattern and an average service life. This procedure eliminates formal calculation, and the estimates are made solely by judgment.

The common methods of computing depreciation require an estimate of service life, and some methods may require an estimate of life expectancy.

Estimates of service life and life expectancy can be computed from a smoothed and extended life table of original life tables developed through life analysis techniques.

Several actuarial techniques are available to construct a life table for depreciation applications, i.e., the individual unit method, original group method, the composite original group, the multiple-original group method, and the annual or retirement rate method.

The construction of a life table usually involves two steps:

1. applying any of the above methods to the survival data
2. graduating the observed life table and fitting the smoothed series to a family of survival functions

Several methods have emerged for the graduation of an observed series. Miller (1946) classified these methods as follows:

1. The graphic method. In this method, the observed values are suitably plotted on graph paper and among them a smooth, continuous curve is drawn as the basis of the graduated series.
2. The interpolation method. In this method, the data are combined into age groups and the graduated series is obtained by interpolation between points determined as representative of the group.
3. The adjusted-average method. In this method, each term of the graduated series is a weighted average of a fixed number of terms of the observed series to which it is central.
4. The difference-equation method. In this method, the graduated series is determined by a difference equation derived from an analytic measure of the relative emphasis to be placed upon fit and smoothness.
5. Graduation by mathematical formula. In this method, the graduated series is represented by a mathematical curve fitted to the data.

Of these methods, the graphic approach and graduation by mathematical formula are the most widely used in the field of depreciation. A commonly used technique of smoothing and of extending the life table is to fit a general linear model, usually a polynomial, to the observed retirement ratios by the least square method. To fit general linear models to retirement ratios, a number of assumptions must be made. One of the objectives of this study is to reexamine the validity of the assumption of independence of retirement ratios.

II. RELATED CONCEPTS

The majority of research that has been associated with actuarial methods can be classified as follows:

1. The investigators find mortality characteristics that better describe the retirement patterns of a property. A mortality law may be expressed as a probability density function $f(x)$ where $f(x)$ is the percentage of units or dollars put in service that are retired during the age of interval x. This is well illustrated by the works of Winfrey and Kurtz (1931), Winfrey (1967), Couch (1957), Kimball (1947), Cowles (1957) and Henderson (1965).

2. The investigators find and/or apply better techniques in which mortality laws of industrial properties are used. The research works of Winfrey (1967), Nichols (1961), Lamp (1968) and White (1968) fall into this classification.

Chiang (1960a) showed the approximate unbiasedness of, and zero correlation between, retirement ratios. The approximate zero covariance property has been used by several researchers (Krane, 1963; Henderson, 1968; Lamp, 1968; and White, 1977) to investigate various methods of fitting that reflect serial independence of disturbance terms.

Due to their importance to the central topic of this study, parts of the works of Chiang (1960a) and White (1977) are briefly presented.

Let n_1 be the total number of units placed in service as a group or vintage at age zero, and n_k be the number of units entering the age interval k. In life studies of physical property, it is assumed that all losses or withdrawals are actual retirements from service. Therefore, the right-censored observations are not considered. Hence, n_k is the number of units exposed to the risk of failure or retirement at the beginning age interval k.

d_k indicates the number of units retired during the k^{th} age interval; $d_k = n_k - n_{k+1}$.

$\hat{q}_k$ denotes the estimated probability of retirement during the k^{th} age interval, conditioned upon exposure to the forces of retirement at the beginning of the k^{th} age interval. By definition

$$\hat{q}_k = \frac{n_k - n_{k+1}}{n_k} = \frac{d_k}{n_k}, \quad k = 1, 2, \ldots, N$$

In depreciation applications, $\hat{q}_k$ is commonly termed a retirement ratio.

$\hat{p}_k$ represents conditional proportion surviving. This is the estimated probability of surviving during the k^{th} age interval, conditioned upon exposure to the retirement at the beginning of the age interval k. By definition,

$$\hat{p}_k = 1 - \hat{q}_k = \frac{n_{k+1}}{n_k}; \quad k = 1, 2, \ldots, N$$

In depreciation applications, $\hat{p}_k$ is called a survival ratio.

q_k indicates the unknown true probability of unit retired in the age interval k.

$p_k = (1 - q_k)$ denotes the unknown true probability of a unit will survive during the age interval k.

$\hat{s}_k$ denotes cumulative proportion surviving. This is an estimate of the probability of surviving to the beginning age interval k. It is given by

$$\hat{s}_k = \hat{p}_{k-1}\,\hat{s}_{k-1} = \frac{n_k}{n_1}; \quad k = 2, 3, \ldots, N - 1.$$

$$= 1.0 \qquad k = 1$$

$$= 0 \qquad k = N$$

The number of units entering the first age-interval (i.e., n_1) can be viewed as n_1 independent trials of a random experiment where each trial can have one of several outcomes. The outcome of a particular unit (trial) may be retirement during the first age-interval, the second age-interval, ..., or the N^{th} age interval. The sum of the number of units retired in all ages is equal to the size of the original vintage put in service. Symbolically,

$$d_1 + d_2 + \ldots + d_N = n_1 .$$

Let θ_k denote the probability that a unit is retired during the k^{th} age interval ($k = 1, 2, \ldots, N$) and $\theta_k = E[\hat{q}_k \; \hat{s}_k]$.

Since a unit is to be retired once and only once somewhere in the life span, then the sum of the probabilities retired in all ages is unity or $\theta_1 + \theta_2 + \ldots + \theta_N = 1$. Thus, we have the well-known lemma 1. The number of units retired, $d_1, \ldots, d_N$ in a life table have a multinomial distribution with the joint probability distribution

$$\Pr[d_1 = \delta_1, \ldots, d_N = \delta_N] = \frac{n_1}{\delta_1! \ldots \delta_k!} \theta_1^{\delta_1} \ldots \theta_N^{\delta_N}; \qquad (2.1)$$

Expectation, variance, and covariance are given, respectively by

$$E(d_k | n_1) = n_1 \theta_k, \text{ for } k = 1, 2, \ldots, N ,$$

$$\text{var}(d_k) = n_1 \theta_k (1 - \theta_k), \text{ for } k = 1, 2, \ldots, N , \qquad (2.2)$$

and

$$\text{cov}(d_k, d_\ell) = -n_1 \theta_k \theta_\ell, \text{ for } k \neq \ell, k, 1 = 1, 2, \ldots N.$$

It follows from (2.2) that expectation, variance and covariance of the unconditional observed proportion of units retired in each age interval, $\frac{d_1}{n_1}, \frac{d_2}{n_1}, \ldots, \frac{d_N}{n_1}$, are given, respectively, by

$$E(\frac{d_k}{n_1}|n_1) = \theta_k, \text{ for } k = 1, 2, \ldots, N,$$

$$var(\frac{d_k}{n_1}) = \frac{\theta_k(1 - \theta_k)}{n_1}, \text{ for } k = 1, 2, \ldots, N, \quad (2.3)$$

$$cov(\frac{d_k}{n_1}, \frac{d_\ell}{n_1}) = \frac{-\theta_k\theta_\ell}{n_1}, \text{ for } k \neq \ell = 1, 2, \ldots, N. \quad (2.4)$$

Lemma 2. The survivors $n_1, n_2, \ldots n_N$ in the life table form a random vector with components having the binomial distribution, and their joint probability function is given by

$$Pr\ (n_1 = n^\circ_1, n_2 = n^\circ_2, \ldots, n_N = n^\circ_N|n_1) =$$

$$\prod_{k=1}^{N} \frac{n^\circ_{k-1}!}{n^\circ_k!(n^\circ_{k-1} - n^\circ_k)!} P_{k-1}^{n^\circ_k}(1 - P_{k-1})^{n^\circ_{k-1} - n^\circ_k}$$

$$\text{for} \quad n^\circ_k = 0, 1, \ldots, n^\circ_{k-1}, \text{ with } n^\circ_o = n_1. \quad (2.5)$$

n_k, the number of units surviving to the beginning of the k^{th} age interval, is a binomial random variable such that

$$E[n_k] = n_1 \sum_{i=k}^{N} \theta_i \quad N_1(1 - \phi_k)$$

$$var(n_k) = n_1(\sum_{i=k}^{N} \theta_i)(\sum_{i=1}^{k-1} \theta_i) \quad (2.6)$$

where

$$\phi_k = \sum_{i=1}^{k-1} \theta_i .$$

Consider the random variable $\hat{q}_k = d_k/n_k$, which is the proportion of those units surviving to the beginning of age interval k that are retired during the k^{th} age interval. It can be shown that an approximate value of the variance of $\hat{q}_k$ is

$$\text{var}(\hat{q}_k) = \frac{q_k(1 - q_k)}{n_1(1 - \sum_{i=1}^{k-1} \theta_i)} . \qquad (2.7)$$

For details of this derivation, see White (1977).

Lemma 3. The conditional observed proportion of units retired, $\hat{q}_k$, (or surviving, $\hat{p}_k$) in an age interval is an unbiased estimator with variance as given by (2.7); the covariance between two proportions $\hat{q}_i$ and $\hat{q}_j$ (or between $\hat{p}_i$ and $\hat{p}_j$) is zero for $i \neq j$; for i, j = 1, 2, ... N.

III. STATEMENT OF OBJECTIVES

Consider the general linear model of retirement ratios,

$$\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$$

where $\underset{\sim}{y}$ is an $N \times 1$ vector of observed retirement ratios, X is a known $(N \times p)$ nonstochastic design matrix of ages, $\underset{\sim}{\beta}$ is a p-dimensional fixed vector of unknown parameter, and $\underset{\sim}{\varepsilon}$ is a $(N \times 1)$ vector of unobservable random error with mean vector $E(\underset{\sim}{\varepsilon}) = 0$ and finite covariance matrix, $\text{cov}(\underset{\sim}{\varepsilon})$.

The structure of $\text{cov}(\underset{\sim}{\varepsilon})$ dictates the method of fitting a linear model to the retirement ratios. For many data generating processes, it is assumed that elements of random error $\underset{\sim}{\varepsilon}$ are identically and independently distributed. Therefore, the covariance matrix is $E(\underset{\sim}{\varepsilon}\ \underset{\sim}{\varepsilon}^{1}) = \sigma^2 I_N$, where the scalar σ^2 is unknown and I_N is a N^{th} order identity matrix. Under a more general formulation, the covariance matrix is represented by $\sigma^2\psi = \Phi$ where ψ is a known positive definite matrix.

This enables the development of a number of estimators for $\underset{\sim}{\beta}$ that depend upon ψ and are good in some sense such as "best linear unbiased." The generalized least square (GLS) estimator is given by

$$\hat{\underset{\sim}{\beta}} = (X^{1}\ \psi^{-1}\ X)^{-1}\ X^{1}\ \psi^{-1}\ \underset{\sim}{y}$$

which depends on ψ and is best linear unbiased.

In practice, the covariance matrix ψ is not given, and is unknown and unobservable, and some restrictive, and hopefully realistic, assumptions are made about its structure. The most common, and the most restrictive, is $\psi = I$. In this case, $\hat{\beta}$ reduces to the least square (LS) estimator,

$$b = (X^1 X)^{-1} X^1 y$$

which depends only on the sample observations.

If the diagonal elements of Φ are not all identical and ε is free from autocorrelation, then Φ can be written as a diagonal matrix with the i^{th} diagonal element given by σ_i^2.

GLS under the general assumption that

$$\Phi = \mathrm{diag}(\sigma_1^2, \ldots \sigma_2^2, \ldots \sigma_N^2)$$

is often referred to as "weighted least square" (WLS).

The covariance structure of $\Phi = \psi\sigma^2$ or $\Phi = \mathrm{diag}(\sigma_1^2, \cdots, \sigma_N^2)$ has two important implications for estimation. The first is that least-squares estimators, while still linear and unbiased (in the case of finite but differing variances), are no longer efficient, no longer providing minimum-variance ("best") estimators among the class of linear unbiased estimators. The second implication is that the estimated variances of the least-square estimators are biased, so the usual tests of statistical significance, such as the students t and F tests, are no longer valid.

As mentioned earlier, vector ε is unobservable. Therefore, its covariance structure must be inferred by an indirect method. One method is to use residuals, e. By finding the relationship between e and ε, the appropriateness of the assumption concerning the covariance of $\underset{\sim}{\varepsilon}$ can be determined. This will not be discussed in this study.

Another way is to examine directly the covariance structure of retirement ratios. This has been adopted for this work.

The specific objectives of this study are as follows:

1. to model the joint continuous and discrete distributions of value or vintage group and life
2. to derive the structures of variance and covariance of retirement ratios of industrial mortality data under whatever mortality law is assumed
3. to reexamine the asymptotic independence among retirement ratios
4. to derive the structures of variances and covariances of retirement ratios when mortality law is assumed to follow geometric distributions

The third objective is very important in conjunction with methods of fitting linear model to retirement ratios and the cost of computer time. The ordinary least square is attributable to its low computational costs, and its support by a broad and sophisticated body of statistical inference.

This study undertook such an investigation which, hopefully, will lead to a better understanding of the correct covariance structure and, hence, to ways of selecting the right method of fitting to the retirement ratios of industrial property.

IV. A MODEL FOR THE JOINT CONTINUOUS DISTRIBUTION OF VALUE AND LIFE

In the field of engineering valuation, most property is measured by dollars rather than physical units. The age at retirement of a physical unit may be independent of the age at retirement of any other physical unit. On the other hand, the physical units comprising a vintage group are often heterogeneous because of their different physical characteristics.

Dollars are homogeneous, and provide a common scale for measuring amounts of property. However, the number of dollars invested in items of a property group generally is not the same as the number of dollars invested in other items of the property group. The age at retirement of one dollar is rarely independent of the age at retirement of some other values. Hence, dollars are not independent random variables. The fact that the ages of retirement of one dollar and some other dollars are not independent leads to the development of bivariate distribution of dollar (value) and age (life).

If there does exist a bivariate distribution of value and life then the relationship between them could be measured by a correlation coefficient.

The coefficient correlation lies between +1 and -1. A correlation +1 or -1 implies that both variables, values and life, are perfectly linearly related. The joint distribution of value and life is then concentrated along the straight line representing that linear relationship.

The joint distribution is bivariate only in the singular sense, and one variable is unessential.

When the coefficient correlation is zero, it follows that both variables are uncorrelated. It is important to note that uncorrelated does not imply independence.

So indeed the bivariate model is a more general one than the model commonly used in previous research. However, the data that support this model may not be available because practical accounting rarely considers the bivariate model. Despite the lack of data that could be used to justify this model, it is theoretically worthwhile to derive the model that may be useful for in future development.

Most previous research has dealt with the univariate distributions such as Iowa Curve, Weibull, Gompertz-Makeham, etc., to describe dollars surviving at any given age. In the univariate case, the surviving dollars (values) or number of units are functions of ages (life), where the ages are fixed random variables.

In the bivariate model, two random variables, i.e., value and life are considered simultaneously. This chapter presents bivariate lognormal and bivariate gamma to describe the proportion of dollars surviving up to a given age.

A. Bivariate Distributions

Let $F_1(x)$ and $F_2(y)$, $f_1(x)$ and $f_2(y)$ be the cumulative probability and density functions of continuous random variables x and y. Then, a bivariate probability function $F(x, y)$ with these marginal distributions is monotonically increasing from zero to unity and it is subject to the following conditions:

1. $F(-\infty, y) = F(x, -\infty) = 0$;

   $F(x, \infty) = F_1(x)$; $F(\infty, y) = F_2(y)$; $F(\infty, \infty) = 1$.

2. The probability content of every rectangle is nonnegative, that is, for every $x_1 < x_2$, $y_1 < y_2$,

$$\Pr(x_1 < x \leq x_2, y_1 < y \leq y_2)$$

$$= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \geq 0 . \tag{4.1}$$

If the second cross partial derivative $\frac{\partial^2 F}{\partial x \partial y}$ exists everywhere, the bivariate distribution has a density $f(x, y)$ equal to its derivative and the condition (4.1) is then equivalent to

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y) \geq 0 .$$

The variables are independent if and only if

$$F(x, y) = F_1(x) F_2(y) .$$

More generally, the marginal density functions $f_1(x)$ and $f_2(y)$ are related to the bivariate density function $f(x, y)$ by

$$\int_{-\infty}^{\infty} f(x, y)\, dy = f_1(x); \int_{-\infty}^{\infty} f(x, y)\, dx = f_2(y) . \tag{4.2}$$

The study of the conditional densities

$$f(x|y) = \frac{f(x, y)}{f_2(y)}; \quad f(y|x) = \frac{f(x, y)}{f_1(x)} \tag{4.3}$$

leads to the conditional expectations $E(x|y)$ and $E(y|x)$ and to the expectation of the cross product

$$E(xy) = \int_{-\infty}^{\infty} y \; E(x|y) \; f_2(y) \; dy \,,$$

and to the classical coefficient correlation

$$\rho = \frac{E(xy) - E(x) \; E(y)}{\sigma_x \sigma_y} . \tag{4.4}$$

B. General Form of F(t)

Consider a bivariate distribution of value and life. Let variable V represent value and variable T denote life (age). Let F(t) be the proportion of total dollars surviving up to age t which is equal to the ratio of dollars surviving at least to the age of t to total dollars initially put in service. In the discrete case, notationally, F(t) can be written as

$$F(t) = \frac{\sum_{i \ni T \geq t} V_i}{\sum_{i=1}^{N} V_i}$$

$$= \frac{\sum_{i \ni T \geq t} V_i/N_t \cdot N_t/N}{\sum_{i=1}^{N} V_i/N}$$

$$= \frac{E(V|T \geq t)\ \bar{F}(t)}{E(V)} \,. \tag{4.5}$$

For the continuous case, F(t) is represented as

$$F(t) = \frac{\int_{-\infty}^{\infty} \dfrac{E(V|S)\ f_T(s)\ ds\ \bar{F}(t)}{\int_t^{\infty} f_T(s)\ ds}}{E(V)}$$

$$F(t) = \frac{\int_t^{\infty} \dfrac{E(V|S)\ f_T(s)\ ds\ \bar{F}(t)}{\bar{F}(t)}}{E(V)}$$

$$= \int_t^{\infty} \frac{E(V|S)\ f_T(s)\ ds}{E(V)} \tag{4.6}$$

where N denotes number of counts of dollars supposedly taking on discrete values.

N_t indicates number of counts of dollars surviving at age t.

$E(V|T \geq t)$ represents the conditional expectation of value given for all ages beyond t.

C. Bivariate Lognormal: The Expression for F(t)

Let the two-dimensional random variable (x, y) have the joint probability density function

$$f_{x,y}(x, y) = f(x, y) = \frac{1}{2\pi\ \sigma_1\ \sigma_2 \sqrt{1 - \rho^2}} \times$$

$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \tag{4.7}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, where σ_1, σ_2, μ_1, μ_2, and ρ are constant such that $-1 < \rho < 1$, $0 < \sigma_1$, $0 < \sigma_2$, $-\infty < \mu_, < \infty$, and $-\infty < \mu_2 < \infty$. Then, the random variable (x, y) is defined to have a bivariate normal distribution.

Bivariate lognormal distribution can be obtained from the bivariate normal distribution by using the following transformations

$$\begin{aligned} V &= e^x \\ S &= e^y \end{aligned} \tag{4.8}$$

where V and S denote random variables corresponding to value (dollar) and life (age), respectively.

The lognormal distribution of V and S are obtained by the following formula:

$$f(v, s) = f_{x,y}(\ln v,\ \ln s)\ |J|\ , \tag{4.9}$$

where $|J|$ is the determinant of the Jacobian of the transformations,

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial s} \end{vmatrix} . \tag{4.10}$$

The derivatives of (4.8) with respect to v and s are

$$\frac{\partial x}{\partial v} = 1/v; \; \frac{\partial x}{\partial s} = 0$$

$$\frac{\partial y}{\partial s} = 0; \; \frac{\partial y}{\partial s} = 1/s .$$

Thus, the determinant of the Jacobian of the transformation,

$$|J| = \begin{vmatrix} 1/v & 0 \\ 0 & 1/s \end{vmatrix} = 1/vs.$$

It follows from formula (4.9) that the bivariate lognormal of value and life can be expressed as

$$f(v, s) = \frac{1}{2\pi \, \sigma_1\sigma_2 \, vs \sqrt{1-\rho^2}}$$

$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\ln v-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(\ln v-\mu_1)(\ln s-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{\ln s-\mu_2}{\sigma_2}\right)^2\right]\right\} \tag{4.11}$$

The marginal and conditional densities $f_1(v)$, $f_2(s)$ and $f(v|s)$, $f(s|v)$ have the form of univariate lognormal.

It can be shown that $f_1(v)$ is univariate lognormal, i.e.,

$$f_1(v) = \frac{1}{v\sigma_1 \sqrt{2\pi}} \exp\left\{ - \frac{(\ln v - \mu_1)^2}{2\sigma_1^2} \right\}.$$

From equation (4.2), the marginal density of V,

$$f_1(v) = \int_{-\infty}^{\infty} f(v, s)\, ds.$$

The substitution of $w = \frac{\ln s - \mu_2}{\sigma_2}$ into (4.11) and upon the completion of the square on w, the marginal density of V can be written as

$$f_1(v) = \int_{-\infty}^{\infty} \frac{1}{2\pi\ \sigma_1 v \sqrt{1 - \rho^2}} \times$$

$$\times \exp\left\{ - \frac{1}{2}\left(\frac{\ln v - \mu_1}{\sigma_1}\right)^2 - \frac{1}{2(1 - \rho^2)} \left(w - \rho\left(\frac{\ln v - \mu_1}{\sigma_1}\right)\right)^2 \right\} dw.$$

Then, the substitutions

$$u = \frac{w - \rho(\ln v - \mu_1)/\sigma_1}{\sqrt{1 - \rho^2}} \quad \text{and} \quad dw = \frac{du}{\sqrt{1 - \rho^2}}$$

show at once that

$$f_1(v) = \frac{1}{v\ \sigma_1 \sqrt{2\pi}} \exp\left\{ - \frac{1}{2}\left(\frac{\ln v - \mu_1}{\sigma_1}\right)^2 \right\} \times$$

$$\times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{1}{2} u^2\right) du.$$

Note that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = 1 .$$

Hence,

$$f_1(v) = \frac{1}{v \sigma_1 \sqrt{2\pi}} \exp\left\{- \frac{(\ln v - \mu_1)^2}{2\sigma_1^2}\right\} .$$

for $0 < \ln v < \infty;$

$0 < \sigma_1; \; 0 < \mu_1 < \infty .$ (4.12)

Similarly, $f_1(s)$ can be shown to be

$$f_2(s) = \frac{1}{s \sigma_2 \sqrt{2\pi}} \exp\left\{- \frac{(\ln s - \mu_2)^2}{2\sigma_2^2}\right\}$$

for $0 < \ln s < \infty; \; 0 < \mu_2 < \infty ;$

$0 < \sigma_2 .$ (4.13)

The conditional density of value given age can be derived as follows. Formula (4.3) gives

$$f(v|s) = \frac{f(v,\ s)}{f(s)}$$

$$= \frac{\dfrac{1}{2\pi\sigma_1\sigma_2 vs\sqrt{1-\rho^2}} \exp\left\{\dfrac{-1}{2(1-\rho^2)}\left[\left(\dfrac{\ln v-\mu_1}{\sigma_1}\right)^2 - 2\rho\,\dfrac{(\ln v-\mu_1)(\ln s-\mu_2)}{\sigma_1\ \sigma_2} + \left(\dfrac{\ln s-\mu_2}{\sigma_2}\right)^2\right]\right\}}{\dfrac{1}{s\sigma_2\sqrt{2\pi}} \exp\left\{-\dfrac{1}{2}\left(\dfrac{\ln s-\mu_2}{\sigma_2}\right)^2\right\}}$$

$$= \frac{1}{v\sigma_1\sqrt{2\pi(1-\rho^2)}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{\ln v-\mu_1}{\sigma_1}\right)^2 - 2\rho\,\frac{(\ln v-\mu_1)(\ln s-\mu_2)}{\sigma_1\ \sigma_2} + \right.\right.$$

$$\left.\left. + (1 - (1-\rho^2))\left(\frac{\ln s - \mu_2}{\sigma_2}\right)^2\right]\right\} \qquad (4.14)$$

Equation (4.14) may be written as

$$f(v|s) = \frac{1}{v\sigma_1\sqrt{2\pi(1-\rho^2)}} \exp\left\{-\frac{1}{2}\left(\frac{\ln v - \mu_1 - \rho\sigma_1/\sigma_2(\ln s - \mu_2)}{\sigma_1\sqrt{1-\rho^2}}\right)^2\right\} \qquad (4.15)$$

Clearly, $f(v|s)$ has the form of lognormal with parameters

$$\mu_1 + \rho\,\sigma_1/\sigma_2(\ln s - \mu_2)\ ,$$

and

$$\sigma_1^{\ 2}(1 - \rho^2)\ .$$

$E(V|S)$ can be deduced by the following relationship. If X is distributed as lognormal,

$$f(x, \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln x - \mu_2}{\sigma}\right)^2\right\}$$

then, the r^{th} moment of X,

$$\mu_r{}^1(x) = \exp(\mu_r + \frac{r^2\sigma^2}{2}). \tag{4.16}$$

For r = 1, (4.16) gives

$$E(x) = e^{\mu+\sigma^2/2} . \tag{4.17}$$

It follows from equation (4.17) that

$$E(V|S) = \exp\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(\ln s - \mu_2) + \frac{\sigma^2(1-\rho^2)}{2}\right)$$

$$= s^{\rho\,\sigma_1/\sigma_2} \exp\left|\mu_1 - \rho\,\sigma_1/\sigma_2\,\mu_2 + \sigma_1{}^2\frac{(1-\rho^2)}{2}\right|. \tag{4.18}$$

Equation (4.6) gives

$$F(t) = \int_t^\infty \frac{E(V|S)\, f_T(s)\, ds}{E(V)} . \tag{4.19}$$

V is distributed as lognormal (4.12). Formula (4.17) gives

$$E(V) = e^{\mu_1+\sigma_1{}^2/2} . \tag{4.20}$$

The numerator of (4.19) is evaluated as follows.

$E(V|S)\ f_T(s)\ ds =$

$$= \exp\left(\mu_1 - \rho\frac{\sigma_1}{\sigma_2}\mu_2 + \frac{\sigma_1^{2}(1-\rho^2)}{2}\right)\int_t^{\infty}\frac{s^{\rho\,\sigma_1\,\sigma_2}}{s\,\sigma_2\sqrt{2\pi}}\exp\left|-\frac{1}{2}\left(\frac{\ln s-\mu_2}{\sigma_2}\right)\right|^2 ds\ . \tag{4.21}$$

Upon the transformation of $\ln s = \zeta$ in the integral of (4.21) yields

$$\int_t^{\infty}\frac{s^{\rho\,\sigma_1/\sigma_2}}{s\,\sigma_2\sqrt{2\pi}}\exp\left|-\frac{1}{2}\left(\frac{\ln s-\mu_2}{\sigma_2}\right)\right|^2 ds =$$

$$= \int_{\ln t}^{\infty}\frac{e^{\rho\,\sigma_1/\sigma_2\,\zeta}}{\sigma_2\sqrt{2\pi}}\exp\left|-\frac{1}{2}\left(\frac{\zeta-\mu_2}{\sigma_2}\right)^2\right| d\zeta$$

$$= \int_{\ln t}^{\infty}\frac{1}{\sigma_2\sqrt{2\pi}}\exp\left|-\frac{(\zeta^2-\zeta(2\mu_2+2\rho\sigma_1\sigma_2)+\mu_2^2)}{2\sigma_2^{\,2}}\right| d\zeta\ . \tag{4.22}$$

Upon the completion of the square of the power of e in the above integrand, the right hand side of (4.22) may be written as

$$\exp\left(\frac{-\mu_2^{\,2}+(\mu_2+\rho\sigma_1\sigma_2)^2}{2\sigma_2^{\,2}}\right)\int_{\ln t}^{\infty}\frac{1}{\sigma_2\sqrt{2\pi}}\exp\left|\frac{-(\zeta-(\mu_2+\rho\sigma_1\sigma_2))^2}{2\sigma_2^{\,2}}\right| d\zeta\ .$$

$$= \exp\left(\mu_2\,\rho\,\sigma_1/\sigma_2 + \rho^2\,\sigma_1^{\,2}/2\right)\left(1-\Phi\left(\frac{\ln t-\mu_2-\rho\,\sigma_1\sigma_2}{\sigma_2}\right)\right)$$

Therefore, $\int_t^\infty E(V|S)\ f_T(s)\ ds =$

$$\exp\ (\mu_1 + \sigma_1^2/2)\left(1 - \Phi\left(\frac{\ln t - \mu_2 - \rho\,\sigma_1\sigma_2}{\sigma_2}\right)\right) \tag{4.23}$$

The substitution of (4.20) and (4.23) into (4.19) yields the proportion of dollars surviving up to age t,

$$F(t) = 1 - \Phi\left(\frac{\ln t - \mu_2 - \rho\,\sigma_1\sigma_2}{\sigma_2}\right).$$

From the definition of coefficient correlation,

$$\rho = \frac{\text{cov}(x, y)}{\text{var}(x) \cdot \text{var}(y)} = \frac{\sigma_{12}}{\sigma_1\sigma_2},$$

gives

$$\rho\ \sigma_1\sigma_2 = \sigma_{12}.$$

Hence,

$$F(t) = 1 - \Phi\left(\frac{\ln t - \mu_2 - \sigma_{12}}{\sigma_2}\right),$$

where

$$\Phi\ (t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\ dz\ . \tag{4.24}$$

The sample estimate of F(t) can be obtained simply by replacing the unknown quantities with the sample quantities,

$$\hat{F}(t) = 1 - \Phi\left(\frac{\ln t - \hat{\mu}_2 - \hat{\sigma}_{12}}{\hat{\sigma}_2}\right), \tag{4.25}$$

where $\hat{\mu}_2$, $\hat{\sigma}_{12}$ and $\hat{\sigma}_2$ are obtained by the maximum likelihood derived from the bivariate normal:

$$\hat{\mu}_2 = \frac{\sum_i t_i}{n}; \quad \hat{\sigma}_{12} = \frac{\sum_i (v_i - \hat{\mu}_1)(t_i - \hat{\mu}_2)}{n};$$

$$\hat{\mu}_1 = \frac{\sum_i v_i}{n}; \quad \hat{\sigma}_1^{\,2} = \frac{\sum_i (v_i - \hat{\mu}_1)^2}{n}; \quad \hat{\sigma}_2 = \frac{\sum_i (t_i - \hat{\mu}_2)^2}{n}.$$

D. Bivariate Gamma: The

Expression for F(t)

Consider random variables

$$X = U + V$$

and (4.26)

$$Y = U + W$$

where U, V and W are independent gamma distributed variables with parameters a, b, and c, respectively, i.e.,

$$f(u) = \frac{1}{\Gamma(a)} u^{a-1} e^{-u},$$

$$f(v) = \frac{1}{\Gamma(b)} v^{b-1} e^{-v},$$

$$f(w) = \frac{1}{\Gamma(c)} w^{c-1} e^{-w}. \tag{4.27}$$

Then, the joint distribution of X and Y,

$$f(x, y) = \frac{e^{-x-y}}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)} \int_0^{\min(x,y)} u^{a-1}\,(x-u)^{b-1}\,(y-u)^{c-1}\,e^{u}\,du \tag{4.28}$$

is called bivariate gamma.

The probability distributions of x and y can be shown, respectively, to be

$$f(x) = \frac{x^{a+b-1}\,e^{-x}}{\Gamma(a+b)}$$

and (4.29)

$$f(y) = \frac{y^{a+c-1}}{\Gamma(a+c)}\,e^{-y}\,.$$

Bivariate gamma distribution of (4.28) can be derived as follows.

Let

$$u = \zeta$$

then

$$\left.\begin{aligned} v &= x - \zeta \\ w &= y - \zeta\,. \end{aligned}\right\} \tag{4.30}$$

The determinant of the Jacobian of transformation of (4.30),

$$|J| = \left|\frac{\partial(u, v, w)}{\partial(\zeta, x, y)}\right| = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 1\,.$$

The joint distribution of u, v, and w can be written as the product of their distributions since they are independent;

$$f(u, v, w) = f(u)\ f(v)\ f(w)$$

$$= \frac{1}{\Gamma(a)\ \Gamma(b)\ \Gamma(c)}\ u^{a-1}\ e^{-u}\ v^{b-1}\ e^{-v}\ w^{c-1}\ e^{-w} . \quad (4.31)$$

The substitution of equation (4.30) into (4.31) yields:

$$f(\zeta, x, y) = \frac{1}{\Gamma(a)\ \Gamma(b)\ \Gamma(c)}\ \zeta^{a-1}\ e^{-\zeta}\ (x-\zeta)^{b-1}\ e^{-(x-\zeta)}\ \times$$

$$\times\ (y-\zeta)^{c-1}\ e^{-(y-\zeta)}$$

or

$$f(\zeta, x, y) = \frac{e^{-x-y}}{\Gamma(a)\ \Gamma(b)\ \Gamma(c)}\ \zeta^{a-1}\ (x-\zeta)^{b-1}\ (y-\zeta)^{c-1}\ e^{\zeta} .$$

The joint distribution of x and y is then obtained by integrating out the above equation with respect to ζ; that is

$$f(x, y) = \frac{e^{-(x+y)}}{\Gamma(a)\ \Gamma(b)\ \Gamma(c)} \int_0^{\min(x,y)} \zeta^{a-1}\ (x-\zeta)^{b-1}\ (y-\zeta)^{c-1}\ e^{\zeta}\ d\zeta . \quad (4.32)$$

The best property of bivariate gamma distribution is the linearity of its regression line, i.e.,

$$E(X|Y) = b + \frac{a}{a+c}\ y . \quad (4.33)$$

The conditional expectation of X given Y, $E(X|Y)$, can be computed as follows.

$$E(X|Y) = E((U+V)|Y)$$

$$= E(U|Y) + E(V|Y) . \quad (4.34)$$

To find $E(U|Y)$ it is necessary to know the joint distribution of Y and U, and, hence, the conditional distribution of U given Y.

Equation (4.26) gives

$$\left.\begin{aligned} & Y = U + W. \\ \text{Let} \quad & U = \eta \quad \text{then} \\ & w = y - \eta. \end{aligned}\right\} \tag{4.35}$$

The determinant of the Jacobian of transformation of (4.35),

$$|J| = \left| \frac{\partial(u, w)}{\partial(\eta, y)} \right| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 .$$

U and W are independent random variables, therefore, their joint distribution can be expressed as the product of their distributions:

$$\begin{aligned} f(u, w) &= f(u)\, f(w) \\ &= \frac{1}{\Gamma(a)\,\Gamma(c)}\, u^{a-1}\, e^{-u}\, w^{c-1}\, e^{-w} . \end{aligned} \tag{4.36}$$

The joint distribution of Y and U can be obtained by substituting equation (4.35) into equation (4.36).

$$\begin{aligned} f(\eta, y) &= f_{u,w}\, (\eta, y)\, |J| \\ &= \frac{1}{\Gamma(a)\,\Gamma(c)}\, \eta^{a-1}\, e^{-\eta}\, (y - \eta)^{c-1}\, e^{-(y-\eta)} \\ &= \frac{1}{\Gamma(a)\,\Gamma(c)}\, \eta^{a-1}\, (y - \eta)^{c-1}\, e^{-y} . \end{aligned} \tag{4.37}$$

The substitution of $\eta = u$ into equation (4.37), will give

$$f(u, y) = \frac{1}{\Gamma(a)\,\Gamma(c)}\, u^{a-1}\, (y - u)^{c-1}\, e^{-y} . \qquad (4.38)$$

The conditional distribution of U given Y,

$$f(u|y) = \frac{f(u, y)}{f(y)}$$

$$= \frac{\frac{1}{\Gamma(a)\,\Gamma(c)}\, u^{a-1}\, (y - u)^{c-1}\, e^{-y}}{\frac{1}{\Gamma(a+c)}\, y^{a+c-1}\, e^{-y}} .$$

$$f(u|y) = \frac{\Gamma(a + c)}{\Gamma(a)\,\Gamma(c)}\, \frac{u^{a-1}\, (y - u)^{c-1}}{y^{a+c-1}} \qquad (4.39)$$

$$E(U|Y) = \int u\, f(u|y)\, du$$

$$= \int u\, \frac{\Gamma(a + c)}{\Gamma(a)\,\Gamma(c)}\, \frac{u^{a-1}\, (y - u)^{c-1}}{y^{a+c-1}}\, du .$$

$$= \frac{\Gamma(a + c)}{\Gamma(a)\,\Gamma(c)} \int \left(\frac{u}{y}\right)^{a} \left(1 - \frac{u}{y}\right)^{c-1} du \qquad (4.40)$$

The substitution of $\zeta = u/y$; $du = y\, d\zeta$, into equation (4.40) results in

$$E(U|Y) = \frac{\Gamma(a + c)}{\Gamma(a)\,\Gamma(c)} \int_0^1 \zeta^{a}\, (1 - \zeta)^{c-1}\, y\, d\zeta$$

$$= y\, \frac{\Gamma(a + c)}{\Gamma(a)\,\Gamma(c)} \int_0^1 \zeta^{a}\, (1 - \zeta)^{c-1}\, d\zeta . \qquad (4.41)$$

This integral is known as the beta function. Hence,

$$E(U|Y) = y\,\frac{\Gamma(a + c)}{\Gamma(a)\,\Gamma(c)}\;B(a + 1, c)$$

$$= y\,\frac{\Gamma(a + c)\;\Gamma(a+1)\;\Gamma(c)}{\Gamma(a)\;\Gamma(c)\;\Gamma(a + c + 1)}$$

$$= \frac{(a)\;\Gamma(a + c)\;\Gamma(a)\;\Gamma(c)}{(a + c)\;\Gamma(a)\;\Gamma(c)\;\Gamma(a+c)}\;y,$$

thus,

$$E(U|Y) = \frac{a}{a + c}\,y\;. \tag{4.42}$$

To evaluate $E(V|Y)$, recall that V and Y are independent. Thus,

$$E(V|Y) = E(V).$$

The distribution of V is a gamma with parameter b, hence,

$$E(V) = b\;. \tag{4.43}$$

The substitution of equations (4.42) and (4.43) into (4.34) results in

$$E(x|y) = b + \frac{a}{a + c}\,y\;.$$

If x and y denote value and life respectively, then formula (4.6) becomes

$$F(t) = \frac{\int_t^\infty E(x|y)\;f_T(y)\;dy}{E(x)}\;. \tag{4.44}$$

The numerator of (4.44) can be evaluated as follows.

$$E(X|Y)\ f_T(y)\ dy =$$

$$\int_t^\infty \left(b + \frac{a}{a+c}\, y\right) \frac{y^{a+c-1}\, e^{-y}}{\Gamma(a+c)}\, dy$$

$$= b \int_t^\infty \frac{y^{a+c-1}\, e^{-y}}{\Gamma(a+c)}\, dy + \frac{a}{a+c} \int_t^\infty \frac{y^{a+c}\, e^{-y}}{\Gamma(a+c)}\, dy$$

$$= b \int_t^\infty \frac{y^{a+c-1}\, e^{-y}}{\Gamma(a+c)}\, dy + a \int_t^\infty \frac{y^{a+c}\, e^{-y}}{\Gamma(a+c+1)}\, dy$$

$$= b(1 - \Gamma_{a+c}(t)) + a(1 - \Gamma_{a+c+1}(t)) \,. \tag{4.45}$$

The denumerator of (4.44) is evaluated next. It follows from (4.29) that variable x is distributed as a gamma distribution with parameter a + b. The standard result gives

$$E(x) = a + b \,. \tag{4.46}$$

The proportion of dollars surviving up to age t is then obtained by substituting equations (4.45) and (4.46) into (4.44); that is,

$$F(t) = \frac{a(1 - \Gamma_{a+c+1}(t)) + b(1 - \Gamma_{a+c}(t))}{a + b} \,, \tag{4.47}$$

where

$$\Gamma_{a+c}(t) = \int_0^t \frac{y^{a+c-1}\, e^{-y}}{\Gamma(a+c)}\, dy .$$

This is known as the cumulative gamma distribution.

V. A MODEL FOR THE JOINT DISCRETE DISTRIBUTION OF VALUE GROUP AND LIFE

Consider a large number, n, of units that are classified into M property groups, with unit values, respectively a_1, a_2, ... a_M. The practical import of the assumption that n is large is the fact that independence can not be expected to hold in the case of property group consisting of a few units. Furthermore, assume value groups a_1, a_2, ..., a_M have different life distributions.

What follows below are the derivations of retirement ratios for the above kind of mortality data, and of the corresponding estimates of the variances and covariances. Under the different mortality characteristics, asymptotic covariances of retirement ratios are generally not zero. When the value-categories do have the same life distribution, it can be shown that asymptotic covariances of retirement ratios are zero.

The models based on these data merely represent mathematical conceptualizations. However, with some modifications these models can be applied to industrial mortality data that are available from the routine accounts of the firm.

The notational and functional relationships introduced here will be used to derive estimates of retirement ratios, and estimates of their variance-covariance structure.

a indicates the cost (price) per unit

n_{a_s} denotes the number of units of value a_s in the property groups under consideration

M represents the number of different value-categories, and hence, the number of distinct life distributions

$n = \sum_{s=1}^{M} n_{a_s}$ denotes the total number of units in the property group under consideration

$\pi_{a_s} = \frac{n_{a_s}}{n}$ is the proportion of units of value a_s in the property group under consideration

$n_{a_{s_j}}$ indicates the number of units of value a_s retired during the age of interval j

$\hat{p}_{a_{s_j}} = \frac{n_{a_{s_j}}}{n_{a_s}}$ is the observed proportion of units of value a_s retired during the age of interval j

$p_{a_{s_j}}$ denotes the true probability that a unit of value a_s retired during the j^{th} age interval under whatever life-distribution is assumed

$$\lambda_j = \sum_{s=1}^{M} a_s \pi_{a_s} p_{a_{s_j}}$$

$$\varepsilon_j = \sum_{s=1}^{M} a_s \pi_{a_s} (\hat{p}_{a_{s_j}} - p_{a_{s_j}})$$

N denotes the number of retirement age intervals

$$\lambda = \sum_{j=1}^{N} \lambda_j = \sum_{k=1}^{N} \sum_{s=1}^{M} a_s \pi_{a_s} p_{a_{s_j}}$$

$$= \sum_{s=1}^{M} \left(\sum_{j=1}^{N} p_{a_{s_j}} \right) a_s \pi_{a_s}$$

$$= \sum_{s=1}^{M} a_s \pi_{a_s}$$

A. The Case of Two-value Category

1. Derivation of observed retirement ratios

To gain a better understanding and increase the ease of computation for the time being, assume that a property group is classified into two distinct values, say a and b.

The retirement ratios are determined as the quotient of the number of units (or dollars) retired during the age interval divided by the number of units (or dollars) surviving at the beginning of that age interval.

r_k = observed retirement ratio for the k^{th} age interval

$$= \frac{\text{dollars retired during the } k^{th} \text{ age interval}}{\text{dollars surviving at the beginning of the } k^{th}}$$

$$= \frac{a\, n_{a_k} + b\, n_{b_k}}{a\, n_a + b\, n_b - \sum_i (a\, n_{a_i} + b\, n_{b_i})} \quad . \tag{5.1}$$

Note here that the subscript 's' is dropped out to facilitate writing the term. Later on this subscript will be needed in considering for the case of the property group having been classified into multivalues.

Both numerator and denumerator of (5.1) can be divided by $n = n_a + n_b$ to give:

$$r_k = \frac{a \cdot \frac{n_a}{n} \cdot \frac{n_{a_k}}{n_a} + b \cdot \frac{n_b}{n} \cdot \frac{n_{b_k}}{n_b}}{a \cdot \frac{n_a}{n} + b \cdot \frac{n_b}{n} - \sum_i \left(a \cdot \frac{n_a}{n} \cdot \frac{n_{a_i}}{n_a} + b \cdot \frac{n_b}{n} \cdot \frac{n_{b_i}}{n_b}\right)} .$$

With the definitions of π's and $\hat{p}$'s as given above, r_k can be expressed as

$$r_k = \frac{a\,\pi_a\,\hat{p}_{a_k} + b\,\pi_b\,\hat{p}_{b_k}}{a\,\pi_a + b\,\pi_b - \sum_i (a\,\pi_a\,\hat{p}_{a_i} + b\,\pi_b\,\hat{p}_{b_i})} . \qquad (5.2)$$

In terms of λ, λ_k and ε_k, (5.2) can be written as

$$r_k = \frac{\lambda_k + \varepsilon_k}{\lambda - \sum_i \lambda_i - \sum_i \varepsilon_i}$$

$$= \frac{\lambda_k + \varepsilon_k}{\lambda - \lambda^\circ_k - \lambda^\circ_k}$$

$$= \frac{\lambda_k}{\lambda - \lambda^\circ_k} \cdot \frac{1 + \varepsilon_k/\lambda_k}{1 - \varepsilon^\circ_k/(\lambda - \lambda^\circ_k)} . \qquad (5.3)$$

Using the Taylor expansion,

$$\frac{1}{1 - \varepsilon^\circ_k(\lambda - \lambda^\circ_k)} = 1 + \frac{\varepsilon^\circ_k}{(\lambda - \lambda^\circ_k)} + \left(\frac{\varepsilon^\circ_k}{\lambda - \lambda^\circ_k}\right)^2 + \ldots \tag{5.4}$$

With the substitution of equation (5.4) into (5.3), $\bar{r}_k$ may be approximated by the linear expression:

$$r_k = \phi_k + \varepsilon_k \cdot \frac{\phi_k}{\lambda_k} + \phi_k \cdot \frac{\varepsilon^\circ_k}{(\lambda - \lambda^\circ_k)} \tag{5.5}$$

where

$$\varepsilon^\circ_k = \sum_{i=1}^{k-1} \lambda_i \; ; \quad \varepsilon^\circ_k = \sum_{i-1}^{k-1} \varepsilon_i \; ; \quad \text{and } \phi_k = \lambda_k/(\lambda - \lambda^\circ_k) \, .$$

2. Derivation of large-sample covariance of r_1 and r_3

The covariance of r_1 and r_3 can be derived as follows. For k = 1 and k = 3, (5.5) gives

$$r_1 = \phi_1 + \varepsilon_1/\lambda$$

and

$$r_3 = \phi_3 + \frac{\varepsilon_3}{(\lambda - \lambda_1 - \lambda_2)} + \lambda_3 \frac{(\varepsilon_1 + \varepsilon_2)}{(\lambda - \lambda_1 - \lambda_2)^2} \; . \; .$$

The covariance becomes:

$$\mathrm{cov}(r_1, r_3) = \frac{\mathrm{cov}(\varepsilon_1, \varepsilon_3)}{\lambda\,(\lambda - \lambda_1 - \lambda_2)} + \lambda_3 \frac{(\mathrm{var}(\varepsilon_1) + \mathrm{cov}(\varepsilon_1, \varepsilon_2))}{(\lambda - \lambda_1 - \lambda_2)^2}$$

$$= 1/\lambda(\lambda - \lambda_1 - \lambda_2)^2\ [(\lambda - \lambda_1 - \lambda_2)\ \mathrm{cov}(\varepsilon_1, \varepsilon_3) +$$

$$+ \lambda_3(\mathrm{var}(\varepsilon_1) + \mathrm{cov}(\varepsilon_1, \varepsilon_2))]\ . \qquad (5.6)$$

According to the lemma 1 (Chiang, 1960a), the number of units retired, in each age interval, from each value category, a_s, have multinomial distributions with parameters n_{a_s}, $p_{a_{s1}}, \ldots, p_{a_{sN}}$. Under the multinomial distributions covariance and variance of ε_j can be evaluated as follows.

$$\mathrm{cov}(\varepsilon_i, \varepsilon_j) = \mathrm{cov}(\sum_i a_s\, \pi_{a_s} (\hat{p}_{a_{si}} - p_{a_{si}}), \sum_j a_s\, \pi_{a_s} (\hat{p}_{a_{sj}} - p_{a_{sj}}))$$

$$= \sum_i \mathrm{cov}(a_s\, \pi_{a_s} (\hat{p}_{a_{si}} - p_{a_{si}}), a_s\, \pi_{a_s} (\hat{p}_{a_{sj}} - p_{a_{sj}})) +$$

$$+ \sum_{s \neq r}\sum \mathrm{cov}(a_s\, \pi_{a_s} (\hat{p}_{a_{si}} - p_{a_{si}}), a_r\, \pi_{a_r} (\hat{p}_{a_{rj}} - p_{a_{rj}}))$$

$$= \sum_s (a_s\, \pi_{a_s})^2\ \mathrm{cov}(\hat{p}_{a_{si}}, \hat{p}_{a_{sj}}) + 0\ .$$

Note that covariance of p's that come from different value groups are zero.

It follows from (2.4) that

$$\mathrm{cov}(\hat{p}_{a_{si}}, \hat{p}_{a_{sj}}) = -\frac{p_{a_{si}}\, p_{a_{sj}}}{n_{a_s}}\ .$$

Hence,

$$\operatorname{cov}(\varepsilon_i, \varepsilon_j) = -\sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{si}} p_{a_{sj}} . \tag{5.7}$$

With the definition of ε_i,

$$\operatorname{var}(\varepsilon_i) = \operatorname{var}(\sum_s a_s \pi_{a_s} (\hat{p}_{a_{si}} - p_{a_{si}}))$$

$$= \sum_s (a_s \pi_{a_s})^2 \operatorname{var}(\hat{p}_{a_{s_i}}) +$$

$$+ \sum_{s \neq r}\sum (a_r \pi_{a_r})(a_s \pi_{a_s}) \operatorname{cov}(\hat{p}_{a_{si}}, \hat{p}_{a_{ri}})$$

$$= \sum_s (a_s \pi_{a_s})^2 \operatorname{var}(\hat{p}_{a_{si}})$$

But

$$\operatorname{var}(\hat{p}_{a_{si}}) = \frac{p_{a_{si}} q_{a_{si}}}{n_{a_s}},$$

Thus,

$$\operatorname{var}(\varepsilon_i) = \sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{si}} q_{a_{si}} . \tag{5.8}$$

For $M = 2$, it follows from (5.7) and (5.8), respectively, that

$$\operatorname{cov}(\varepsilon_i, \varepsilon_j) = -\frac{(a\, \pi_a)^2 p_{a_i} q_{a_j}}{n_a} - \frac{(b\, \pi_b)^2 p_{b_i} q_{b_j}}{n_b}$$

and

$$\mathrm{var}(\varepsilon_j) = + \frac{(a\ \pi_a)^2\ p_{a_j}\ q_{a_j}}{n_a} + \frac{(b\ \pi_b)^2\ p_{b_j}\ q_{b_j}}{n_b}\ .$$

The substitution of λ, λ_i's, and variance-covariance of ε's into (5.6) yields

$$\begin{aligned}
\mathrm{cov}(r_1,\ r_3) = {} & (a\ \pi_a + b\ \pi_b)^{-1} \times \\
& \times\ (a\ \pi_a + b\ \pi_b - a\ \pi_a\ p_{a_1} - b\ \pi_b\ p_{b_1} - a\ \pi_a\ p_{a_2} - b\ \pi_b\ p_{b_2})^{-2} \times \\
& \times\ [(a\ \pi_a + b\ \pi_b - a\ \pi_a\ p_{a_1} - b\ \pi_b\ p_{b_1} - a\ \pi_a\ p_{a_2} - b\ \pi_b\ p_{b_2}) \times \\
& \times\ (-\ (a\ \pi_a)^2 \frac{p_{a_1}\ p_{a_3}}{n_a} - (b\ \pi_b)^2 \frac{p_{b_1} p_{b_3}}{n_b}) + (a\ \pi_a\ p_{a_3} + b\ \pi_b\ p_{b_3}) \times \\
& \times\ (\frac{(a\ \pi_a)^2}{n_a}\ p_{a_1}\ q_{a_1} + \frac{(b\ \pi_b)^2}{n_b}\ p_{b_1}\ q_{b_1} - \frac{(a\ \pi_a)^2\ p_{a_1}\ p_{a_2}}{n_a} - \\
& -\ \frac{(b\ \pi_b)^2\ p_{b_1}\ p_{b_2}}{n_b})]\ .
\end{aligned} \tag{5.9}$$

After the terms in the numerator of (5.9) are multiplied out, covariance of r_1 and r_3 becomes:

$$\text{cov}(r_1, r_3) = (a\,\pi_a + b\,\pi_b)^{-1} \times$$

$$\times (a\,\pi_a + b\,\pi_b - a\,\pi_a\,p_{a_1} - b\,\pi_b\,p_{b_1} - a\,\pi_a\,p_{a_2} - b\,\pi_b\,p_{b_2})^{-2} \times$$

$$\times [n_a^{-1}((a\,\pi_a)^3\,p_{a_1}^{\,2}\,p_{a_3} + (a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}\,p_{a_3}\,p_{b_1} +$$

$$+ (a\,\pi_a)^3\,p_{a_1}\,p_{a_3}\,p_{a_2} + (a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}\,p_{a_3}\,p_{b_2} -$$

$$- (a\,\pi_a)^3\,p_{a_1}\,p_{a_3} - (a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}\,p_{a_3} +$$

$$+ (a\,\pi_a)^3\,p_{a_1}\,q_{a_1}\,p_{a_3} + (a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}\,q_{a_1}\,p_{b_3} -$$

$$- (a\,\pi_a)^3\,p_{a_1}\,p_{a_2}\,p_{a_3} - (a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}\,p_{a_2}\,p_{b_3}) +$$

$$+ n_b^{-1}((a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}\,p_{b_3}\,p_{a_1} + (b\,\pi_b)^3\,p_{b_1}^{\,2}\,p_{b_3} +$$

$$+ (a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}\,p_{b_3}\,p_{a_2} + (b\,\pi_b)^3\,p_{b_1}\,p_{b_3}\,p_{b_2} -$$

$$- (a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}\,p_{b_3} - (b\,\pi_b)^3\,p_{b_1}\,p_{b_3} +$$

$$+ (a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}\,q_{b_1}\,p_{a_3} + (b\,\pi_b)^3\,p_{b_1}\,q_{b_1}\,p_{b_3} -$$

$$- (a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}\,p_{b_2}\,p_{a_3} - (b\,\pi_b)^3\,p_{b_1}\,p_{b_2}\,p_{b_3})] \,. \qquad (5.10)$$

The substitution of $q_{a_1} = 1 - p_{a_1}$ and $q_{b_1} = 1 - p_{b_1}$ into (5.10), along with the further simplification the $\mathrm{cov}(r_1, r_3)$ may be expressed as

$$\mathrm{cov}(r_1, r_3) = (a\,\pi_a + b\,\pi_b)^{-1} \times$$

$$\times\ (a\,\pi_a + b\,\pi_b - a\,\pi_a \sum_{i=1}^{2} p_{a_i} - b\,\pi_b \sum_{i=1}^{2} p_{b_i})^{-2} \times$$

$$\times\ [n_a^{-1}((a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}(p_{b_3} - p_{a_3}) +$$

$$+\ (a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}(p_{a_3}\,p_{b_2} - p_{a_3}\,p_{b_3}) +$$

$$+\ (a\,\pi_a)^2\,(b\,\pi_b)\,p_{a_1}(p_{a_3}\,p_{b_1} - p_{a_1}\,p_{b_3})) +$$

$$+\ n_b^{-1}((a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}(p_{a_3} - p_{b_3}) +$$

$$+\ (a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}(p_{b_3}\,p_{a_2} - p_{b_2}\,p_{a_3}) +$$

$$+\ (a\,\pi_a)(b\,\pi_b)^2\,p_{b_1}(p_{a_1}\,p_{b_3} - p_{a_3}\,p_{b_1}))]\,. \qquad (5.11)$$

If the life distributions of the value groups a and b are identical, $p_{a_i} = p_{b_i} \equiv p_i$ for $i = 1, 2, 3, \ldots, N$, then it can be easily observed that

$$\mathrm{cov}(r_1, r_3) = 0.$$

3. Derivation of large-sample covariance of r_2 and r_3

Recall that the assumption that the property group is categorized into two value groups, say a and b, has been made. It follows from (5.5) for k = 2 and k = 3 that

$$r_2 = \frac{\lambda_1}{(\lambda - \lambda_1)} + \frac{\lambda_2}{(\lambda - \lambda_1)} + \lambda_2 \frac{\varepsilon_1}{(\lambda - \lambda_1)^2}$$

and

$$r_3 = \frac{\lambda_3}{(\lambda - \lambda_1 - \lambda_2)} + \frac{\varepsilon_3}{(\lambda - \lambda_1 - \lambda_2)} + \lambda_3 \frac{(\varepsilon_1 + \varepsilon_2)}{(\lambda - \lambda_1 - \lambda_2)^2} .$$

With the use of the definition of covariance, covariance of r_2 and r_3 can be written as

$$\begin{aligned} \mathrm{cov}(r_2, r_3) = {} & (\lambda - \lambda_1)^{-2} (\lambda - \lambda_1 - \lambda_2)^{-2} \times \\ & \times [(\lambda_3(\lambda - \lambda_1) + \lambda_2\lambda_3) \mathrm{cov}(\varepsilon_1, \varepsilon_2) + \\ & + (\lambda - \lambda_1)(\lambda - \lambda_1 - \lambda_2) \mathrm{cov}(\varepsilon_2, \varepsilon_3) + \\ & + \lambda_2(\lambda - \lambda_1 - \lambda_2) \mathrm{cov}(\varepsilon_1, \varepsilon_3) + \\ & + \lambda_3(\lambda - \lambda_1) \mathrm{var}(\varepsilon_2) + \lambda_2\lambda_3 \mathrm{var}(\varepsilon_1)] . \end{aligned} \tag{5.12}$$

Covariances and variances of ε's in (5.12) can be evaluated by (5.7) and (5.8), respectively.

The substitution of λ, λ_j's, variances and covariances of ε's into (5.12), permits the writing of the covariance of r_2 and r_3 as

$$\mathrm{cov}(r_2, r_3) = (a\,\pi_a + b\,\pi_b - a\,\pi_a\,p_{a_1} - b\,\pi_b\,p_{b_1})^{-2} \times$$

$$\times\,(a\,\pi_a + b\,\pi_b - \sum_{i=1}^{2}(a\,\pi_a\,p_{a_i} + b\,\pi_b\,p_{b_i}))^{-2} \times$$

$$\times\,[(-(a\,\pi_a)^2\,\frac{p_{a_1}\,p_{a_2}}{n_a} - (b\,\pi_b)^2\,\frac{p_{b_1}\,p_{b_2}}{n_b})((a\,\pi_a\,p_{a_3} + b\,\pi_b\,p_{b_3}) \times$$

$$\times\,(a\,\pi_a + b\,\pi_b - a\,\pi_a\,p_{a_1} - b\,\pi_b\,p_{b_1}) +$$

$$+\,(a\,\pi_a\,p_{a_2} + b\,\pi_b\,p_{b_2})(a\,\pi_a\,p_{a_3} + b\,\pi_b\,p_{b_3})) +$$

$$+\,(-(a\,\pi_a)^2\,\frac{p_{a_2}\,p_{a_3}}{n_a} - (b\,\pi_b)^2\,\frac{p_{b_2}\,p_{b_3}}{n_b}) \times$$

$$\times\,(a\,\pi_a + b\,\pi_b - a\,\pi_a\,p_{a_1} - b\,\pi_b\,p_{b_1}) \times$$

$$\times\,(a\,\pi_a + b\,\pi_b - a\,\pi_a\,p_{a_1} - b\,\pi_b\,p_{b_1} - a\,\pi_a\,p_{a_2} - b\,\pi_b\,p_{b_2}) +$$

$$+\,(-(a\,\pi_a)^2\,\frac{p_{a_1}\,p_{a_3}}{n_a} - (b\,\pi_b)^2\,\frac{p_{b_1}\,p_{b_3}}{n_b}) \times$$

$$\times\,(a\,\pi_a\,p_{a_2} + b\,\pi_b\,p_{b_2}) \times$$

$$\times\,(a\,\pi_a + b\,\pi_b - a\,\pi_a\,p_{a_1} - b\,\pi_b\,p_{b_1} - a\,\pi_a\,p_{a_2} - b\,\pi_b\,p_{b_2}) +$$

$$+ \left((a\ \pi_a)^2 \frac{p_{a_2}\ q_{a_2}}{n_a} + (b\ \pi_b)^2 \frac{p_{b_2}\ q_{b_2}}{n_b}\right) \times$$

$$\times (a\ \pi_a\ p_{a_3} + b\ \pi_b\ p_{b_3})(a\ \pi_a + b\ \pi_b - a\ \pi_a\ p_{a_1} - b\ \pi_b\ p_{b_1}) +$$

$$+ \left((a\ \pi_a)^2 \frac{p_{a_1}\ q_{a_1}}{n_a} + (b\ \pi_b)^2 \frac{p_{b_1}\ q_{b_1}}{n_b}\right) \times$$

$$\times (a\ \pi_a\ p_{a_2} + b\ \pi_b\ p_{b_2})(a\ \pi_a\ p_{a_3} + b\ \pi_b\ p_{b_3})]\ . \qquad (5.13)$$

The multiplication of the terms in the numerator of (5.13) yields

$$\mathrm{cov}(r_2,\ r_3) = (a\ \pi_a + b\ \pi_b - a\ \pi_a\ p_{a_1} - b\ \pi_b\ p_{b_1})^{-2} \times$$

$$\times (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_{i=1}^{2} p_{a_i} - b\ \pi_b \sum_{i=1}^{2} p_{b_i}))^{-2} \times$$

$$\times [n_a^{-1}(-(a\ \pi_a)^4\ p_{a_1}\ p_{a_2} p_{a_3} - (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ p_{a_2}\ p_{a_3} +$$

$$+ (a\ \pi_a)^4\ p_{a_1}^{\ 2}\ p_{a_2}\ p_{a_3} + (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ p_{a_2}\ p_{a_3}\ p_{b_1} -$$

$$- (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_1}\ p_{a_2}\ p_{b_3} - (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1}\ p_{a_2}\ p_{b_3} +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}^{\ 2}\ p_{a_2}\ p_{b_3} + (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1} p_{a_2} p_{b_1} p_{b_3} -$$

$$- (a\ \pi_a)^4\ p_{a_1}\ p_{a_2}^{\ 2}\ p_{a_3} - (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ p_{a_2}^{\ 2}\ p_{b_3} -$$

$$- (a\ \pi_a)^3 (b\ \pi_b) p_{a_1} p_{a_2} p_{a_3} p_{b_2} - (a\ \pi_a)^2 (b\ \pi_b)^2 p_{a_1} p_{a_2} p_{b_2} p_{b_3} -$$

$$- (a\ \pi_a)^4 p_{a_2} p_{a_3} - 2(a\ \pi_a)^3 (b\ \pi_b) p_{a_2} p_{a_3} + (a\ \pi_a)^2 (b\ \pi_b)^2 p_{a_2} p_{a_3} +$$

$$+ (a\ \pi_a)^4 p_{a_2} p_{a_3} p_{a_1} + (a\ \pi_a)^3 (b\ \pi_b) p_{a_2} p_{a_3} p_{b_1} +$$

$$+ (a\ \pi_a)^4 p_{a_2}^{\ 2} p_{a_3} + (a\ \pi_a)^3 (b\ \pi_b) p_{a_2} p_{a_3} p_{b_2} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b) p_{a_1} p_{a_2} p_{a_3} + (a\ \pi_a)^2 (b\ \pi_b)^2 p_{a_2} p_{a_3} p_{b_1} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b) p_{a_2}^{\ 2} p_{a_3} + (a\ \pi_a)^2 (b\ \pi_b)^2 p_{a_2} p_{a_3} p_{b_2} +$$

$$+ (a\ \pi_a)^4 p_{a_2} p_{a_3} p_{a_1} + (a\ \pi_a)^3 (b\ \pi_b) p_{a_2} p_{a_3} p_{a_1} -$$

$$- (a\ \pi_a)^4 p_{a_1}^{\ 2} p_{a_2} p_{a_3} - (a\ \pi_a)^3 (b\ \pi_b) p_{a_2} p_{a_3} p_{a_1} p_{b_1} -$$

$$- (a\ \pi_a)^4 p_{a_2}^{\ 2} p_{a_3} p_{a_1} - (a\ \pi_a)^3 (b\ \pi_b) p_{a_2} p_{a_3} p_{a_1} p_{b_2} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b) p_{a_2} p_{a_3} p_{b_1} + (a\ \pi_a)^2 (b\ \pi_b)^2 p_{a_2} p_{a_3} p_{b_1} -$$

$$- (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_2}\ p_{a_3}\ p_{a_1}\ p_{b_1} - (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_2}\ p_{a_3}\ p_{b_1}^{\ 2} -$$

$$- (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_2}^{\ 2}\ p_{a_3}\ p_{b_1} - (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_2}\ p_{a_3}\ p_{b_1}\ p_{b_2} -$$

$$- (a\ \pi_a)^4\ p_{a_1}\ p_{a_3}\ p_{a_2} - (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_1}\ p_{a_3}\ p_{a_2} +$$

$$+ (a\ \pi_a)^4\ p_{a_1}^{\ 2}\ p_{a_2}\ p_{a_3} + (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_1}\ p_{a_3}\ p_{a_2}\ p_{b_1} +$$

$$+ (a\ \pi_a)^4\ p_{a_1}\ p_{a_3}\ p_{a_2}^{\ 2} + (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_1}\ p_{a_3}\ p_{a_2}\ p_{b_2} -$$

$$- (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_1}\ p_{a_3}\ p_{b_2} - (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_1}\ p_{a_3}\ p_{b_2} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_1}^{\ 2}\ p_{a_3}\ p_{b_2} + (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_1}\ p_{a_3}\ p_{b_1}\ p_{b_2} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_1}\ p_{a_3}\ p_{a_2}\ p_{b_2} + (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_1}\ p_{a_3}\ p_{b_2}^{\ 2} +$$

$$+ (a\ \pi_a)^4\ p_{a_2}\ q_{a_2}\ p_{a_3} + (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_2}\ q_{a_2}\ p_{a_3} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_2}\ q_{a_2}\ p_{b_3} + (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_2}\ q_{a_2}\ p_{b_3} -$$

$$- (a\ \pi_a)^4\ p_{a_2}\ q_{a_2}\ p_{a_1}\ p_{a_3} - (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_2}\ q_{a_2}\ p_{a_3}\ p_{b_1} -$$

$$- (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_2}\ q_{a_2}\ p_{a_1}\ p_{b_3} - (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_2}\ q_{a_2}\ p_{b_1}\ p_{b_3} +$$

$$+ (a\ \pi_a)^4\ p_{a_1}\ q_{a_1}\ p_{a_2}\ p_{a_3} + (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ q_{a_1}\ p_{a_2}\ p_{b_3} +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ q_{a_1}\ p_{a_3}\ p_{b_2} + (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1}\ q_{a_1}\ p_{b_2}\ p_{b_3}) +$$

$$+ n_b^{-1}(-(a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{b_2}\ p_{a_3} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}\ p_{a_3} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{b_2}\ p_{a_1}\ p_{a_3} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}\ p_{a_3}\ p_{b_1} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}\ p_{b_3} - (b\ \pi_b)^4\ p_{b_1}\ p_{b_2}\ p_{b_3} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}\ p_{a_1}\ p_{b_3} + (b\ \pi_b)^4\ p_{b_1}^{\ 2}\ p_{b_2}\ p_{b_3} -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{b_2}\ p_{a_2}\ p_{a_3} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}\ p_{a_2}\ p_{b_3} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}\ p_{a_3}\ p_{b_2} - (b\ \pi_b)^4\ p_{b_1}\ p_{b_2}\ p_{b_2}\ p_{b_3} -$$

$$- (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{b_2}\ p_{b_3} - 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3} - (b\ \pi_b)^4\ p_{b_2}\ p_{b_3} +$$

$$+ (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{b_2}\ p_{b_3}\ p_{a_1} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{b_1} +$$

$$+ (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{b_2}\ p_{b_3}\ p_{a_2} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}^{\ 2}\ p_{b_3} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{a_1} + (b\ \pi_b)^4\ p_{b_2}\ p_{b_3}\ p_{b_1} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{a_2} + (b\ \pi_b)^4\ p_{b_2}^{\ 2}\ p_{b_3} +$$

$$+ (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{b_2}\ p_{b_3}\ p_{a_1} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{a_1} -$$

$$- (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{b_2}\ p_{b_3}\ p_{a_1}^{\ 2} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{a_1}\ p_{b_1} -$$

$$- (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{b_2}\ p_{b_3}\ p_{a_1}\ p_{a_2} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}^{\ 2}\ p_{b_3}\ p_{a_1} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{b_1} + (b\ \pi_b)^4\ p_{b_2}\ p_{b_3}\ p_{b_1} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{a_1}\ p_{b_1} - (b\ \pi_b)^4\ p_{b_2}\ p_{b_3}\ p_{b_1}^{\ 2} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}\ p_{a_2}\ p_{b_1} - (b\ \pi_b)^4\ p_{b_2}^{\ 2}\ p_{b_3}\ p_{b_1} -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{b_3}\ p_{a_2} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_3}\ p_{a_2} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{b_3}\ p_{a_1}\ p_{a_2} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_3}\ p_{a_2}\ p_{b_1} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{b_3}\ p_{a_2}^{\ 2} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_3}\ p_{a_2}\ p_{b_2} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_3}\ p_{b_2} - (b\ \pi_b)^4\ p_{b_1}\ p_{b_3}\ p_{b_2} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_3}\ p_{a_1}\ p_{b_2} + (b\ \pi_b)^4\ p_{b_1}^{\ 2}\ p_{b_2}\ p_{b_3} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_3}\ p_{a_2}\ p_{b_2} + (b\ \pi_b)^4\ p_{b_1}\ p_{b_2}^{\ 2}\ p_{b_3} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_2}\ q_{b_2}\ p_{a_3} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ q_{b_2}\ p_{a_3} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ q_{b_2}\ p_{b_3} + (b\ \pi_b)^4\ p_{b_2}\ q_{b_2}\ p_{b_3} -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_2}\ q_{b_2}\ p_{a_1}\ p_{a_3} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ q_{b_2}\ p_{a_3}\ p_{b_1} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ q_{b_2}\ p_{a_1}\ p_{b_3} - (b\ \pi_b)^4\ p_{b_2}\ q_{b_2}\ p_{b_1}\ p_{b_3} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ q_{b_1}\ p_{a_2}\ p_{a_3} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ q_{b_1}\ p_{a_2}\ p_{b_3} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ q_{b_1}\ p_{a_3}\ p_{b_2} + (b\ \pi_b)^4\ p_{b_1}\ q_{b_1}\ p_{b_2}\ p_{b_3})]. \tag{5.14}$$

After extensive simplification, (5.14) may be presented by

$$\text{cov}(r_2, r_3) = (a\ \pi_a + b\ \pi_b - a\ \pi_a\ p_{a_1} - b\ \pi_b\ p_{b_1})^{-2} \times$$

$$\times (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_{i=1}^{2} p_{a_i} - b\ \pi_b \sum_{i=1}^{2} p_{b_i})^{-2} \times$$

$$\times [n_a^{-1}((a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_2}(p_{a_3}\ p_{b_1} - p_{a_1}\ p_{b_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_2}\ p_{b_1}(p_{a_1}\ p_{b_3} - p_{a_3}\ p_{b_1}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1}\ p_{b_2}(p_{a_3}\ p_{b_2} - p_{a_2}\ p_{b_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_2}(p_{b_3} - p_{a_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_2}(p_{a_3}\ p_{b_2} - p_{a_2}\ p_{b_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_2}\ p_{b_1}(p_{a_3} - p_{b_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_2}\ p_{b_1}(p_{a_2}\ p_{b_3} - p_{a_3}\ p_{b_2}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1}\ p_{b_2}(p_{b_3} - p_{a_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1}\ p_{b_2}(p_{a_3}\ p_{b_1} - p_{a_1}\ p_{b_3}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ p_{a_2}(p_{b_3} - p_{a_3}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ p_{a_2}(p_{a_3} - p_{b_3}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ p_{a_2}(p_{a_1}\ p_{b_3} - p_{a_3}\ p_{b_1}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_2}(p_{b_3} - p_{a_3}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_2}\ p_{a_3}(p_{b_2} - p_{a_2}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_2}^{\ 2}(p_{a_3} - p_{b_3}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_2}(p_{a_3}\ p_{b_1} - p_{a_1}\ p_{b_3}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_1}\ p_{a_2}(p_{a_3}\ p_{b_1} - p_{a_1}\ p_{b_3})) +$$

$$+ n_b^{-1}((a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_2}(p_{a_1}\ p_{b_3} - p_{b_1}\ p_{a_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1}\ p_{b_2}(p_{a_3}\ p_{b_1} - p_{b_3}\ p_{a_1}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{a_2}(p_{b_3}\ p_{a_2} - p_{b_2}\ p_{a_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_2}(p_{a_3} - p_{b_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_2}(p_{b_3}\ p_{a_2} - p_{b_2}\ p_{a_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_2}\ p_{a_1}(p_{b_3} - p_{a_3}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_1}\ p_{b_2}(p_{b_2}\ p_{a_3} - p_{b_3}\ p_{a_2}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_1}\ p_{a_2}(p_{a_3} - p_{b_3}) +$$

$$+ (a\ \pi_a)\ (b\ \pi_b)^3\ p_{b_1}\ p_{a_2}(p_{b_3}\ p_{a_1} - p_{a_3}\ p_{b_1}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}(p_{a_3} - p_{b_3}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}(p_{b_3} - p_{a_3}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}(p_{a_3}\ p_{b_1} - p_{b_3}\ p_{a_1}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}(p_{a_3} - p_{b_3}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}\ p_{b_3}(p_{a_2} - p_{b_2}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}{}^2(p_{b_3} - p_{a_3}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_2}(p_{b_3}\ p_{a_1} - p_{a_3}\ p_{b_1}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_1}\ p_{b_2}(p_{b_3}\ p_{a_1} - p_{a_3}\ p_{b_1}))]\ . \tag{5.15}$$

When the property of values a and b are respectively subjected to the same mortality law, i.e., $p_{a_i} = p_{b_i} \equiv p_i$ for all i, then all the terms in the square bracket of (5.15) cancel out; therefore, $\text{cov}(r_1, r_3) = 0$.

4. Derivation of large-sample covariance of r_k and r_ℓ

The covariance of r_k and r_ℓ where $k < \ell$ can be derived as follows. For k = k and k = ℓ, (5.5) gives

$$r_k = \phi_k + \frac{\varepsilon_k}{(\lambda - \sum_i \lambda_i)} + \lambda_k \frac{(\sum_i \varepsilon_i)}{(\lambda - \sum_i \lambda_i)^2}$$

and

$$r_\ell = \phi_\ell + \frac{\varepsilon_\ell}{(\lambda - \sum_j \lambda_j)} + \lambda_\ell \frac{(\sum_j \varepsilon_j)}{(\lambda - \sum_j \lambda_j)^2}\ .$$

From the definition of covariance it follows that

$$\operatorname{cov}(r_k, r_\ell) = \frac{\operatorname{cov}(\varepsilon_k, \varepsilon_\ell)}{(\lambda - \sum_i \lambda_i)(\lambda - \sum_j \lambda_j)} + \frac{\lambda_\ell \operatorname{cov}(\varepsilon_k, \sum_j \varepsilon_j)}{(\lambda - \sum_i \lambda_i)(\lambda - \sum_j \lambda_j)^2}$$

$$+ \frac{\lambda_k \operatorname{cov}(\varepsilon_\ell, \sum_i \varepsilon_i)}{(\lambda - \sum_i \lambda_i)^2 (\lambda - \sum_j \lambda_j)} + \frac{\lambda_k \lambda_\ell \operatorname{cov}(\sum_i \varepsilon_i, \sum_j \varepsilon_j)}{(\lambda - \sum_i \lambda_i)^2 (\lambda - \sum_j \lambda_j)^2} .$$

The covariance of r_k and r_ℓ may be written as

$$\operatorname{cov}(r_k, r_\ell) = (\lambda - \sum_i \lambda_i)^{-2} (\lambda - \sum_j \lambda_j)^{-2} \times$$

$$\times \; [(\lambda - \sum_i \lambda_i)(\lambda - \sum_j \lambda j) \cdot \operatorname{cov}(\varepsilon_k, \varepsilon_\ell) + \lambda_\ell (\lambda - \sum_i \lambda_i) \times$$

$$\times \; (\operatorname{var}(\varepsilon_k) + \sum_{j \neq k} \operatorname{cov}(\varepsilon_k, \varepsilon_j)) + \lambda_k (\lambda - \sum \lambda_j) \sum_i \operatorname{cov}(\varepsilon_\ell, \varepsilon_i) +$$

$$+ \; \lambda_k \lambda_\ell (\sum_i \operatorname{var}(\varepsilon_i) + \sum_{i,j \neq i} \operatorname{cov}(\varepsilon_i, \varepsilon_j))] \tag{5.16}$$

for $i = 1, 2, \ldots, k-1$; $j = 1, 2, \ldots, \ell-1$.

Under the assumption that the groups of property of values a and b are respectively subject to the multimonial distributions, covariances and variances of ε's in (5.16) can be computed by (5.7) and (5.8), respectively.

The substitution of λ, λ_i's and variances and covariances of ε's into (5.16), permits the covariance of r_k and r_ℓ to be written as

$$\mathrm{cov}(r_k, r_\ell) = (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i})^{-2} \times$$

$$\times (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_j p_{a_j} - b\ \pi_b \sum_j p_{b_j})^{-2} \times$$

$$\times [(a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i}) \times$$

$$\times (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_j p_{a_j} - b\ \pi_b \sum_j p_{b_j}) \times$$

$$\times (- \frac{(a\ \pi_a)^2}{n_a} p_{a_k} p_{a_\ell} - \frac{(b\ \pi_b)^2}{n_b} p_{b_k} p_{b_\ell}) + (a\ \pi_a p_{a_\ell} + b\ \pi_b p_{b_\ell}) +$$

$$+ (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i}) \times$$

$$\times (\frac{(a\ \pi_a)^2}{n_a} p_{a_k} q_{a_k} + \frac{(b\ \pi_b)^2}{n_a} p_{b_k} q_{b_k} -$$

$$- \frac{(a\ \pi_a)^2}{n_a} p_{a_k} \sum_{j \neq k} p_{a_j} - \frac{(b\ \pi_b)^2}{n_b} p_{b_k} \sum_{j \neq k} p_{a_j}) +$$

$$+ (a\ \pi_a\ p_{a_k} + b\ \pi_b\ p_{b_k}) \times$$

$$\times (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_j p_{a_j} - b\ \pi_b \sum_j p_{b_j}) \times$$

$$\times (- \frac{(a\ \pi_a)^2\ p_{a_\ell}}{n_a} \sum_i p_{a_i} - \frac{(b\ \pi_b)^2\ p_{b_\ell}}{n_b} \sum_i p_{b_i}) +$$

$$+ \quad (a\ \pi_a\ p_{a_k} + b\ \pi_b\ p_{b_k})(a\ \pi_a\ p_{a_\ell} + b\pi_b\ p_{b_\ell}) \times$$

$$\times \left(\frac{(a\ \pi_a)^2}{n_a} \sum_i p_{a_i}\ q_{a_i} + \frac{(b\ \pi_b)^2}{n_b} \sum_i p_{b_i}\ q_{b_i} - \right.$$

$$\left. - \frac{(a\ \pi_a)^2}{n_a} \sum_{i\neq j} p_{a_i}\ p_{a_j} - \frac{(b\ \pi_b)^2}{n_b} \sum_{i\neq j} p_{b_i}\ p_{b_j}\right)]. \qquad (5.17)$$

for $i = 1, 2, \ldots, k-1$; $j = 1, 2, \ldots, \ell-1$; $k < \ell$.

After the terms in the numerator of (5.17) are multiplied out, (5.17) can be written as

$$\mathrm{cov}(r_k, r_\ell) = (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i})^{-2} \times$$

$$\times (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_j p_{a_j} - b\ \pi_b \sum_j p_{b_j})^{-2} \times$$

$$\times [(n_a^{-1}(-(a\ \pi_a)^4\ p_{a_k}\ p_{a_\ell} - 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ p_{a_\ell} +$$

$$+ (a\ \pi_a)^4\ p_{a_k}\ p_{a_\ell} \sum_j p_{a_j} + (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_j p_{b_j} +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_j p_{a_j} + (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ p_{a_\ell} \sum_j p_{b_j} +$$

$$+ (a\ \pi_a)^4\ p_{a_k}\ p_{a_\ell} \sum_i p_{a_i} + (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_i p_{b_i} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_i p_{a_i} + (a\ \pi_a)^2 (b\ \pi_b)^2 p_{a_k} p_{a_\ell} \sum_i p_{b_i} -$$

$$- (a\ \pi_a)^4\ p_{a_k}\ p_{a_\ell} \sum_i \sum_j p_{a_i}\ p_{a_j} - (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k} p_{a_\ell} \sum_i \sum_j p_{a_i}\ p_{b_j} -$$

$$- (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_i \sum_j p_{a_j}\ p_{b_i} -$$

$$- (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_k}\ p_{a_\ell} \sum_i \sum_j p_{b_i}\ p_{b_j} -$$

$$- (a\ \pi_a)^4\ p_{a_k}\ p_{a_\ell} \sum_{j \neq k} p_{a_j} - (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ p_{b_\ell} \sum_{j \neq k} p_{a_j} -$$

$$- (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_{j \neq k} p_{a_j} - (a\ \pi_a)^2 (b\ \pi_b)^2 p_{a_k} p_{b_\ell} \sum_{j \neq k} p_{a_j} +$$

$$+ (a\ \pi_a)^4\ p_{a_k}\ p_{a_\ell} \sum_i \sum_{j \neq k} p_{a_i}\ p_{a_j} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_i \sum_{j \neq k} p_{b_i}\ p_{a_j} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ p_{b_\ell} \sum_i \sum_{j \neq k} p_{a_i}\ p_{a_j} +$$

$$+ (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_k}\ p_{b_\ell} \sum_i \sum_{j \neq k} p_{b_i}\ p_{a_j} +$$

$$+ (a\ \pi_a)^4\ p_{a_k}\ q_{a_k}\ p_{a_\ell} + (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ q_{a_k}\ p_{b_\ell} +$$

$$+ (a\ \pi_a)^3 (b\ \pi_b)\ p_{a_k}\ q_{a_k}\ p_{a_\ell} + (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_k}\ q_{a_k}\ p_{b_\ell} -$$

$$- (a\,\pi_a)^4\, p_{a_k}\, q_{a_k}\, p_{a_\ell} \sum_i p_{a_i} - (a\,\pi_a)^3\, (b\,\pi_b)\, p_{a_k}\, q_{a_k}\, p_{a_\ell} \sum_i p_{b_i} -$$

$$- (a\,\pi_a)^3\, (b\,\pi_b)\, p_{a_k}\, q_{a_k}\, p_{b_\ell} \sum_i p_{a_i} -$$

$$- (a\,\pi_a)^2\, (b\,\pi_b)^2\, p_{a_k}\, q_{a_k}\, p_{b_\ell} \sum_i p_{b_i} -$$

$$- (a\,\pi_a)^4\, p_{a_k}\, p_{a_\ell} \sum_i p_{a_i} - (a\,\pi_a)^3\, (b\pi_b)\, p_{b_k}\, p_{a_\ell} \sum_i p_{a_i} -$$

$$- (a\,\pi_a)^3\, (b\,\pi_b)\, p_{a_k}\, p_{a_\ell} \sum_i p_{a_i} - (a\,\pi_a)^2\, (b\,\pi_b)\, p_{a_\ell}\, p_{b_k} \sum_i p_{a_i} +$$

$$+ (a\,\pi_a)^4\, p_{a_k}\, p_{a_\ell} \sum_i \sum_j p_{a_i}\, p_{a_j} + (a\,\pi_a)^3\, (b\,\pi_b)\, p_{a_k}\, p_{a_\ell} \sum_i \sum_j p_{a_i}\, p_{b_j} +$$

$$+ (a\,\pi_a)^3\, (b\,\pi_b)\, p_{a_\ell}\, p_{b_k} \sum_i \sum_j p_{a_i}\, p_{a_j} +$$

$$+ (a\,\pi_a)^2\, (b\,\pi_b)^2\, p_{a_\ell}\, p_{b_k} \sum_i \sum_j p_{a_i}\, p_{b_j} +$$

$$+ (a\,\pi_a)^4\, p_{a_k}\, p_{a_\ell} \sum_i p_{a_i}\, q_{a_i} + (a\,\pi_a)^3\, (b\,\pi_b)\, p_{a_k}\, p_{b_\ell} \sum_i p_{a_i}\, q_{a_i} +$$

$$+ (a\,\pi_a)^3\, (b\,\pi_b)\, p_{b_k}\, p_{a_\ell} \sum_i p_{a_i}\, q_{a_i} +$$

$$+ (a\,\pi_a)^2\, (b\,\pi_b)\, p_{b_k}\, p_{b_\ell} \sum_i p_{a_i}\, q_{a_i} -$$

$$- (a\,\pi_a)^4\, p_{a_k}\, p_{a_\ell} \sum_{i \neq j} \sum p_{a_i}\, p_{a_j} - (a\,\pi_a)^3\, (b\,\pi_b)\, p_{a_k}\, p_{b_\ell} \sum_{i \neq j} \sum p_{a_i}\, p_{a_j} -$$

$$- (a\ \pi_a)^3\ (b\ \pi_b)\ p_{b_k}\ p_{a_\ell} \sum_{i \neq j} p_{a_i}\ p_{a_j}\ -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ p_{b_\ell} \sum_{i \neq j} p_{a_i}\ p_{a_j}) +$$

$$+ n_b^{-1}(-(a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ p_{b_\ell}\ -$$

$$- 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} - (a\ \pi_a)^4\ p_{b_k}\ p_{b_\ell} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ p_{b_\ell} \sum_j p_{a_j} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_j p_{b_j} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_j p_{a_j} + (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_j p_{b_j} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ p_{b_\ell} \sum_i p_{a_i} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k} p_{b_\ell} \sum_i p_{b_i} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_i p_{a_i} + (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_i p_{b_i}\ -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ p_{b_\ell} \sum_i \sum_j p_{a_i}\ p_{a_j}\ -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_i \sum_j p_{a_i}\ p_{b_j}\ -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_i \sum_j p_{b_i}\ p_{a_j} - (b\ \pi_b)^4\ p_{b_k} p_{b_\ell} \sum_i \sum_j p_{b_i}\ p_{b_j}\ -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ p_{a_\ell} \sum_{j \neq k} p_{b_j} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k} p_{b_\ell} \sum_{j \neq k} p_{b_j} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{a_\ell} \sum_{j\neq k} p_{b_j} - (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_{j\neq k} p_{b_j} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ p_{a_\ell} \sum_i \sum_{j\neq k} p_{a_i}\ p_{b_j} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{a_\ell} \sum_i \sum_{j\neq k} p_{b_i}\ p_{b_j} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_i \sum_{j\neq k} p_{a_i}\ p_{b_j} +$$

$$+ (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_i \sum_{j\neq k} p_{b_i}\ p_{b_j} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ q_{b_k}\ p_{a_\ell} + (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b\ell}\ q_{b_k} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ q_{b_k}\ p_{a_\ell} + (b\ \pi_b)^4\ p_{b_k}\ q_{b_k}\ p_{b_\ell} -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ q_{b_k}\ p_{a_\ell} \sum_i p_{a_i} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k} q_{b_k} p_{a_\ell} \sum_i p_{b_i} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ q_{b_k}\ p_{b_\ell} \sum_i p_{a_i} - (b\ \pi_b)^4\ p_{b_k}\ q_{b_k}\ p_{b_\ell} \sum_i p_{b_i} -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ p_{b_\ell} \sum_i p_{b_i} - (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_i p_{b_i} -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_\ell} \sum_i p_{b_i} - (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_i p_{b_i} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ p_{b_\ell} \sum_i \sum_j p_{b_i}\ p_{a_j} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_\ell} \sum_i \sum_j p_{b_i}\ p_{b_j}\ +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{b_\ell} \sum_i \sum_j p_{b_i}\ p_{a_j} + (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_i \sum_j p_{b_i}\ p_{b_j}\ +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ p_{a_\ell} \sum_i p_{b_i}\ q_{b_i} + (a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_\ell} \sum_i p_{b_i}\ q_{b_i} +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{a_\ell} \sum_i p_{b_i}\ q_{b_i} + (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_i p_{b_i}\ q_{b_i}\ -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)\ p_{a_k}\ p_{a_\ell} \sum_{i \neq j} \sum p_{b_i}\ p_{b_j}\ -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_\ell} \sum_{i \neq j} \sum p_{b_i}\ p_{b_j}\ -$$

$$- (a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ p_{a_\ell} \sum_{i \neq j} \sum p_{b_i}\ p_{b_j}\ -$$

$$- (b\ \pi_b)^4\ p_{b_k}\ p_{b_\ell} \sum_{i \neq j} \sum p_{b_i}\ p_{b_j})] \tag{5.18}$$

for $i - 1, 2, \ldots, k-1;$

$j = 1, 2, \ldots, \ell-1;$

$k < \ell.$

After extensive simplification, (5.18) may be expressed as

$$\text{cov}(r_k,\ r_\ell) = (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i})^{-2}\ \times$$

$$\times (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_j p_{a_j} - b\ \pi_b \sum_j p_{b_j})^{-2} \times$$

$$\times [n_a^{-1}((a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}(p_{a_\ell} \sum_j p_{b_j} - p_{b_\ell} \sum_j p_{a_j}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_\ell}(p_{a_k} \sum_i p_{b_i} - p_{b_k} \sum_i p_{a_i}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_\ell}(p_{a_k} \sum_i p_{b_i} - p_{b_k} \sum_i p_{a_i}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_\ell}(\sum_j p_{a_j})(p_{b_k} \sum_i p_{a_i} - p_{a_k} \sum_i p_{b_i}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_\ell}(\sum_j p_{b_j})(p_{b_k} \sum_i p_{a_i} - p_{a_k} \sum_i p_{b_i}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_\ell}(\sum_j p_{a_j})(p_{a_k} \sum_i p_{b_i} - p_{b_k} \sum_i p_{a_i}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_\ell}(\sum_j p_{a_j})(p_{a_k} \sum_i p_{b_i} - p_{b_k} \sum_i p_{a_i}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_\ell}(p_{b_k} \sum_i p_{a_i} - p_{a_k} \sum_i p_{b_i}) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}(p_{b_\ell} - p_{a_\ell}) + (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}(p_{b_\ell} - p_{a_\ell}))+$$

$$+ n_b^{-1}((a\ \pi_a)^2\ (b\ \pi_b)^2 p_{b_k}(p_{b_\ell} \sum_j p_{a_j} - p_{a_\ell} \sum_j p_{b_j}) +$$

$$+ (a\ \pi_a)(b\ \pi_b)^3\ p_{b_\ell}(p_{b_k} \sum_i p_{a_i} - p_{a_k} \sum_i p_{b_i}) +$$

$$+ (a\,\pi_a)^2\,(b\,\pi_b)^2\,p_{b_\ell}(p_{b_k}\sum_i p_{a_i} - p_{a_k}\sum_i p_{b_i}) +$$

$$+ (a\,\pi_a)(b\,\pi_b)^3\,p_{b_\ell}(\sum_j p_{b_j})(p_{a_k}\sum_i p_{b_i} - p_{b_k}\sum_i p_{a_i}) +$$

$$+ (a\,\pi_a)^2\,(b\,\pi_b)^2\,p_{b_\ell}(\sum_j p_{a_j})(p_{a_k}\sum_i p_{b_i} - p_{b_k}\sum_i p_{a_i}) +$$

$$+ (a\,\pi_a)(b\,\pi_b)^3\,p_{b_\ell}(\sum_j p_{b_j})(p_{b_k}\sum_i p_{a_i} - p_{a_k}\sum_i p_{b_i}) +$$

$$+ (a\,\pi_a)^2\,(b\,\pi_b)^2\,p_{a_\ell}(\sum_j p_{b_j})(p_{a_k}\sum_i p_{b_i} - p_{b_k}\sum_i p_{a_i}) +$$

$$+ (a\,\pi_a)(b\,\pi_b)^3\,p_{b_\ell}(p_{a_k}\sum_i p_{b_i} - p_{b_k}\sum_i p_{a_i}) +$$

$$+ (a\,\pi_a)(b\,\pi_b)^3\,p_{b_k}(p_{a_\ell} - p_{b_\ell}) + (a\,\pi_a)^2\,(b\,\pi_b)^2\,p_{b_k}(p_{a_\ell} - p_{b_\ell}))] . \tag{5.19}$$

It is important to note here that $\mathrm{cov}(r_k, r_\ell)$ for $k < \ell$, and $k, \ell = 1, 2, \ldots, N$, is a symmetric function for values a and b. It means that $\mathrm{cov}(r_k, r_\ell)$ can be written as

$$\mathrm{cov}(r_k, r_\ell) \equiv f(n_a, \pi_a, p_a, n_b, \pi_b, p_b) \equiv$$

$$f(n_b, \pi_b, p_b, n_a, \pi_a, p_a).$$

When the value groups of a and b have the same life distribution, i.e.,

$$p_{a_i} = p_{b_i} \equiv p_i \text{ for all } i.$$

then, it is easily observed that $\text{cov}(r_k, r_\ell) = 0$.

5. Derivation of large-sample variance of r_k

The variance of the retirement ratio for each age interval can be computed by the following formula. For $k = \ell$, (5.16) gives

$$\text{var}(r_k) = (\lambda - \sum_i \lambda_i)^{-4} \, [(\lambda - \sum_i \lambda_i)^2 \, \text{var}(\varepsilon_k) +$$

$$+ 2\lambda_k(\lambda - \sum_i \lambda_i) \sum_i \text{cov}(\varepsilon_k, \varepsilon_i) +$$

$$+ \lambda_k^2(\sum_i \text{var}(\varepsilon_i) + \sum_{i \neq j}\sum \text{cov}(\varepsilon_i, \varepsilon_j))] \tag{5.20}$$

for $i, j = 1, 2, \ldots, k-1$.

Covariance and variance of ε's in the square bracket of (5.20) can be computed by formulas (5.7) and (5.8), respectively.

The substitution of λ, λ_i's, variances and covariance of ε's into (5.20) yields

$$\text{var}(r_k) = (a\,\pi_a + b\,\pi_b - a\,\pi_a \sum_i p_{a_i} - b\,\pi_b \sum_i p_{b_i})^{-4} \times$$

$$\times [(a\,\pi_a + b\,\pi_b - a\,\pi_a \sum_i p_{a_i} - b\,\pi_b \sum_i p_{b_i})^2 \times$$

$$\times \left(\frac{(a\,\pi_a)^2}{n_a} p_{a_k} q_{a_k} + \frac{(b\,\pi_b)^2}{n_b} p_{b_k} q_{b_k} \right) +$$

$$+ 2(a\ \pi_a\ p_{a_k} + b\ \pi_b\ p_{b_k})(a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i})\times$$

$$\times (-\frac{(a\ \pi_a)^2}{n_a} p_{a_k} \sum_i p_{a_i} - \frac{(b\ \pi_b)^2}{n_b} p_{b_k} \sum_i p_{b_i}) +$$

$$+ (a\ \pi_a\ p_{a_k} + b\ \pi_b\ p_{b_k})^2 (\frac{(a\ \pi_a)^2}{n_a} \sum_i p_{a_i}\ q_{a_i} + \frac{(b\ \pi_b)^2}{n_b} \sum_i p_{b_i}\ q_{b_i} -$$

$$- \frac{(a\ \pi_a)^2}{n_a} \sum_{i \neq j}\sum p_{a_i}\ p_{a_j} - \frac{(b\ \pi_b)^2}{n_b} \sum_{i \neq j}\sum p_{b_i}\ p_{b_j})] \qquad (5.21)$$

for i, j = 1, 2, ..., k-1.

The multiplication of the terms in the numerator of (5.21) gives

$$\text{var}(r_k) = (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i}))^{-4} \times$$

$$\times [n_a^{-1}((a\ \pi_a)^4\ p_{a_k}\ q_{a_k} + 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ q_{a_k} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ q_{a_k} -$$

$$- 2(a\ \pi_a)^4\ p_{a_k}\ q_{a_k} \sum_i p_{a_i} - 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ q_{a_k} \sum_i p_{b_i} -$$

$$- 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ q_{a_k} \sum_i p_{a_i} - 2(a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ q_{a_k} \sum_i p_{b_i} +$$

$$+ (a\ \pi_a)^4\ p_{a_k}\ q_{a_k} \sum_i \sum_j p_{a_i}\ p_{a_j} + (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ q_{a_k} \sum_i \sum_j p_{b_i}\ p_{b_j} +$$

$$+ 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ q_{a_k} \sum_i \sum_j p_{a_i}\ p_{b_j} -$$

$$- 2(a\ \pi_a)^4\ p_{a_k}^{\ 2} \sum_i p_{a_i} - 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}^{\ 2} \sum_i p_{a_i} +$$

$$+ 2(a\ \pi_a)^4\ p_{a_k}^{\ 2} \sum_i \sum_j p_{a_i}\ p_{a_j} + 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}^{\ 2} \sum_i \sum_j p_{a_i}\ p_{b_j} -$$

$$- 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{b_k} \sum_i p_{a_i} - 2(a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ p_{b_k} \sum_i p_{a_i} +$$

$$+ 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{b_k} \sum_i \sum_j p_{a_i}\ p_{a_j} +$$

$$+ 2(a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}\ p_{b_k} \sum_i \sum_j p_{a_i}\ p_{b_j} +$$

$$+ (a\ \pi_a)^4\ p_{a_k}^{\ 2} \sum_i p_{a_i}\ q_{a_i} + 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{b_k} \sum_i p_{a_i}\ q_{a_i} +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}^{\ 2} \sum_i p_{a_i}\ q_{a_i} - (a\ \pi_a)^4\ p_{a_k}^{\ 2} \sum\sum_{i \neq j} p_{a_i}\ p_{a_j} +$$

$$+ 2(a\ \pi_a)^3\ (b\ \pi_b)\ p_{a_k}\ p_{b_k} \sum_{i \neq j} p_{a_i}\ p_{a_j} -$$

$$- (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}^{\ 2} \sum\sum_{i \neq j} p_{a_i}\ p_{a_j}\ ) +$$

$$+ n_b^{-1}((a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ q_{b_k} + 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ q_{b_k} + (b\ \pi_b)^4\ p_{b_k}\ q_{b_k} -$$

$$- 2(a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}\ q_{b_k} \sum_i p_{a_i} - 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ q_{b_k} \sum_i p_{b_i} -$$

$$- 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ q_{b_k} \sum_i p_{a_i} - 2(b\ \pi_b)^4\ p_{b_k}\ q_{b_k} \sum_i p_{b_i} +$$

$$+ (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{b_k}\ q_{b_k} \sum_i \sum_j p_{a_i}\ p_{a_j} + (b\ \pi_b)^4\ p_{b_k}\ q_{b_k} \sum_i \sum_j p_{b_i}\ p_{b_j} +$$

$$+ 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ q_{b_k} \sum_i \sum_j p_{a_i}\ p_{b_j} -$$

$$- 2(a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_k}\ p_{b_k} \sum_i p_{b_i} - 2(a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_k} \sum_i p_{b_i} +$$

$$+ 2(a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_k}\ p_{b_k} \sum_i \sum_j p_{b_i}\ p_{a_j} +$$

$$+ 2(a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_k} \sum_i \sum_j p_{b_i}\ p_{b_j} -$$

$$- 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}^2 \sum_i p_{b_i} - 2(b\ \pi_b)^4\ p_{b_k}^2 \sum_i p_{b_i} +$$

$$+ 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}^2 \sum_i \sum_j p_{b_i}\ p_{a_j} + 2(b\ \pi_b)^4\ p_{b_k}^2 \sum_i \sum_j p_{b_i}\ p_{b_j} +$$

$$+ (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_k}^2 \sum_i p_{b_i}\ q_{b_i} + 2(a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_k} \sum_i p_{b_i}\ q_{b_i} +$$

$$+ (b\ \pi_b)^4\ p_{b_k}^2 \sum_i p_{b_i}\ q_{b_i} - (a\ \pi_a)^2 (b\ \pi_b)^2\ p_{a_k}^2 \sum_{i \neq j} \sum p_{b_i}\ p_{b_j} -$$

$$- 2(a\ \pi_a)(b\ \pi_b)^3\ p_{a_k}\ p_{b_k} \sum_{i \neq j} \sum p_{b_i}\ p_{b_j} - (b\ \pi_b)^4\ p_{b_k}^2 \sum_{i \neq j} \sum p_{b_i}\ p_{b_j})]. \tag{5.22}$$

Again, with considerable simplification, (5.22) can be written as

$$\text{var}(r_k) = (a\ \pi_a + b\ \pi_b - a\ \pi_a \sum_i p_{a_i} - b\ \pi_b \sum_i p_{b_i})^{-4} \times$$

$$\times [n_a^{-1}((a\,\pi_a)^4 (p_{a_k} q_{a_k} - 2p_{a_k} \sum_i p_{a_i} + p_{a_k}^{\;2} \sum_i p_{a_i} + p_{a_k} \sum_i \sum_j p_{a_i} p_{a_j}) +$$

$$+ (a\,\pi_a)^3 (b\,\pi_b)(2p_{a_k} q_{a_k} - 2p_{a_k} \sum_i p_{a_i} - 2p_{a_k} \sum_i p_{b_i} +$$

$$+ 2p_{a_k} p_{b_k} \sum_i p_{a_i} + 2p_{a_k} \sum_i \sum_j p_{a_i} p_{b_j}) +$$

$$+ (a\,\pi_a)^2 (b\,\pi_b)^2 (p_{a_k} q_{a_k} - 2p_{a_k} \sum_i p_{b_i} + p_{b_k}^{\;2} \sum_i p_{a_i} +$$

$$+ p_{a_k} \sum_i \sum_j p_{b_i} p_{b_j}) +$$

$$+ 2(a\,\pi_a)^3 (b\,\pi_b)\, p_{a_k}(p_{a_k} \sum_i p_{b_i} - p_{b_k} \sum_i p_{a_i}) +$$

$$+ 2(a\,\pi_a)^2 (b\,\pi_b)^2\, p_{a_k}(p_{a_k} \sum_i p_{b_i} - p_{b_k} \sum_i p_{a_i}) +$$

$$+ (a\,\pi_a)^2 (b\,\pi_b)^2\, p_{a_k}(\sum_j p_{b_j})(p_{b_k} \sum_i p_{a_i} - p_{a_k} \sum_i p_{b_i}) +$$

$$+ (a\,\pi_a)^2 (b\,\pi_b)^2\, p_{b_k}(\sum_i p_{a_i})(p_{a_k} \sum_j p_{b_j} - p_{b_k} \sum_j p_{a_j})) +$$

$$+ n_b^{-1}((b\,\pi_b)^4 (p_{b_k} q_{b_k} - 2p_{b_k} \sum_i p_{b_i} + p_{b_k}^{\;2} \sum_i p_{b_i} +$$

$$+ p_{b_k} \sum_i \sum_j p_{b_i} p_{b_j}) +$$

$$+ (a\,\pi_a)(b\,\pi_b)^3 (2p_{b_k} q_{b_k} - 2p_{b_k} \sum_i p_{b_i} - 2p_{b_k} \sum_i p_{a_i} +$$

$$+ 2p_{a_k} p_{b_k} \sum_i p_{b_i} + 2p_{b_k} \sum_i \sum_j p_{a_i} p_{b_j}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ (p_{b_k}\ q_{b_k} - 2p_{b_k} \sum_i p_{a_i} + p_{a_k}{}^2 \sum_i p_{b_i} +$$

$$+ p_{b_k} \sum_i \sum_j p_{a_i}\ p_{a_j}) +$$

$$+ 2(a\ \pi_a)(b\ \pi_b)^3\ p_{b_k}\ (p_{b_k} \sum_i p_{a_i} - p_{a_k} \sum_i p_{b_i}) +$$

$$+ 2(a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}(p_{b_k} \sum_i p_{a_i} - p_{a_k} \sum_i p_{b_i}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{a_k}(\sum_i p_{b_i})(p_{b_k} \sum_j p_{a_j} - p_{a_k} \sum_j p_{b_j}) +$$

$$+ (a\ \pi_a)^2\ (b\ \pi_b)^2\ p_{b_k}(\sum_j p_{a_j})(p_{a_k} \sum_i p_{b_i} - p_{b_k} \sum_i p_{a_i}))]. \qquad (5.23)$$

Of special interest here is the case in which all value groups die according to the same mortality characteristic, i.e.,

$$p_{a_i} = p_{b_i} \equiv p_i \text{ for all } i.$$

Then, equation (5.23) can be simplified:

$$\text{var}(r_k) = ((a\ \pi_a + b\ \pi_b)(1 - \sum_i p_i))^{-4} \times$$

$$\times [n_a^{-1}((a\ \pi_a)^4\ (p_k\ q_k - 2p_k \sum_i p_i + p_k{}^2 \sum_i p_i + p_k \sum_i \sum_j p_i\ p_j) +$$

$$+ (a\ \pi_a)^3\ (b\ \pi_b)(2p_k\ q_k - 2p_k \sum_i p_i - 2p_k \sum_i p_i +$$

$$+ 2p_k^2 \sum_i p_i + 2p_k \sum_i \sum_j p_i p_j) +$$

$$+ (a\,\pi_a)^2 (b\,\pi_b)^2 (p_k q_k - 2p_k \sum_i p_i + p_k^2 \sum_i p_i + p_k \sum_i \sum_j p_i p_j) +$$

$$+ n_b^{-1}((b\,\pi_b)^4 (p_k q_k - 2p_k \sum_i p_i + p_k^2 \sum_i p_i + p_k \sum_i \sum_j p_i p_j) +$$

$$+ (a\,\pi_a)(b\,\pi_b)^3 (2p_k q_k - 2p_k \sum_i p_i - 2p_k \sum_i p_i + 2p_k^2 \sum_i p_i +$$

$$+ 2p_k \sum_i \sum_j p_i p_j) + (a\,\pi_a)^2 (b\,\pi_b)^2 (p_k q_k - 2p_k \sum_i p_i +$$

$$+ p_k^2 \sum_i p_i + p_k \sum_i \sum_j p_i p_j)]. \qquad (5.24)$$

Equation (5.24) can be further simplified to become

$$\mathrm{var}(r_k) = (a\,\pi_a + b\,\pi_b)^{-4} \times$$

$$\times [n_a^{-1}((a\,\pi_a)^4 + 2(a\,\pi_a)^3 (b\,\pi_b) + (a\,\pi_a)^2 (b\,\pi_b)^2) +$$

$$+ n_b^{-1}((b\,\pi_b)^4 + 2(a\,\pi_a)(b\,\pi_b)^3 + (b\,\pi_b)^2 (a\,\pi_a)^2)]$$

$$\times \frac{(R_k - R_k^2)}{(1 - \sum_{i=1}^{k-1} p_i)}$$

$$= (a\,\pi_a + b\,\pi_b)^{-2} (n_a^{-1}(a\,\pi_a)^2 + n_b^{-1}(b\,\pi_b)^2) \frac{(R_k - R_k^2)}{(1 - \sum_{i=1}^{k-1} p_i)} \qquad (5.25)$$

$$\text{where } R_k = \frac{P_k}{(1 - \sum_{i=1}^{k-1} p_i)} . \tag{5.26}$$

This is known as the population (true) retirement ratios for the k^{th} age interval. It is of interest that the variance of r_k's is increasing in R_k for $0 < R_k \leq 0.5$ and decreasing for $0.5 < R_k \leq 1$.

B. The Case of Several Value Categories

The model for multi-value category is essentially an extension of the model for two-value classes. When a property group is classified into more than two-value groups, it becomes a model for multi-values. The practical example of this model may be thought of as property groups consisting of several vintages that were installed in successive years. The younger vintages of similar property may have different values (costs) than the older ones. Inflation, technological change, etc., may be responsible for the units of similar property having different values.

1. Derivation of observed retirement ratios

In a manner similar to that used in (5.1), the retirement ratios for the k^{th} age interval can be written as

$$r_k = \frac{\sum_{s=1}^{M} a_s n_{a_{sk}}}{\sum_{s=1}^{M} a_s n_{a_s} - \sum_{i=1}^{k-1} \sum_{s=1}^{M} n_{a_{si}}} \tag{5.27}$$

and as before, (5.27) can be expressed as

$$r_k = \frac{\sum_s a_s \pi_{a_s} p_{a_{sk}}}{\sum_s a_s \pi_{a_s} - \sum_i \sum_s a_s \pi_{a_s} \hat{p}_{a_{si}}} . \tag{5.28}$$

In terms of λ, λ_k and ε_k, (5.28) can be written exactly as (5.3).

Hence, r_k can be approximated by linear order terms, i.e.,

$$r_k = \phi_k + \varepsilon_k \frac{\phi_k}{\lambda_k} + \varepsilon^\circ{}_k \frac{\phi_k}{(\lambda - \lambda^\circ{}_k)} . \tag{5.29}$$

2. Derivation of large-sample covariance of r_k and r_ℓ

It follows from (5.29) that

$$r_k = \phi_k + \varepsilon_k \frac{\phi_k}{\lambda_k} + \varepsilon^\circ{}_k \frac{\phi_k}{(\lambda - \lambda^\circ{}_k)}$$

and

$$r_\ell = \phi_\ell + \varepsilon_\ell \frac{\phi_\ell}{\lambda_\ell} + \varepsilon^\circ{}_\ell \frac{\phi_\ell}{(\lambda - \lambda^\circ{}_\ell)} .$$

Therefore, the covariance of r_k and r_ℓ can be exactly written as (5.16), i.e.,

$$\mathrm{cov}(r_k, r_\ell) = (\lambda - \sum_i \lambda_i)^{-2} (\lambda - \sum_j \lambda_j)^{-2} \times$$

$$\times [(\lambda - \sum_i \lambda_i)(\lambda - \sum_j \lambda_j) \mathrm{cov}(\varepsilon_k, \varepsilon_\ell) +$$

$$+ \lambda_\ell(\lambda - \sum_i \lambda_i)(\mathrm{var}(\varepsilon_k) + \sum_{j \neq k} \mathrm{cov}(\varepsilon_k, \varepsilon_j)) +$$

$$+ \lambda_k(\lambda - \sum_j \lambda_j)(\sum_i \mathrm{cov}(\varepsilon_\ell, \varepsilon_i)) +$$

$$+ \lambda_k \lambda_\ell(\sum_i \mathrm{var}(\varepsilon_i) + \sum_{i \neq j}\sum \mathrm{cov}(\varepsilon_i, \varepsilon_j))] \, . \qquad (5.30)$$

And, the covariance and variance of ε's of (5.30) can be evaluated by (5.7) and (5.8), respectively. The substitution of λ, λ_i's, variances and covariances of ε_i's into (5.30), gives the covariance of r_k and r_ℓ as

$$\mathrm{cov}(r_k, r_\ell) =$$

$$= (\sum_s a_s \pi_{a_s} - \sum_i \sum_r a_r \pi_{a_r} p_{a_{ri}})^{-2} (\sum_u a_u \pi_{a_u} - \sum_j \sum_v a_v \pi_{a_v} p_{a_{vj}})^{-2} \times$$

$$\times [-(\sum_r a_r \pi_{a_r} - \sum_i \sum_v a_v \pi_{a_v} p_{a_{vi}})(\sum_u a_u \pi_{a_u} - \sum_j \sum_w a_w \pi_{a_w} p_{a_{wj}}) \times$$

$$\times (\sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{sk}} p_{a_{s\ell}}) + (\sum_r a_r \pi_{a_r} p_{a_{r\ell}})(\sum_u a_u \pi a_u -$$

$$- \sum_i \sum_v a_v \pi_{a_v} p_{a_{vi}})(\sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{sk}} q_{a_{sk}} - \sum_{j \neq k} \sum \frac{(a_s \pi_{a_s})^2}{n_{a_s}} \times$$

$$\times p_{a_{sk}} p_{a_{sj}}) + (\sum_r a_r \pi_{a_r} p_{a_{rk}})(\sum_u a_u \pi_{a_u} - \sum_j \sum_v a_v \pi_{a_v} p_{a_{vj}}) \times$$

$$\times (\sum_i \sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{s\ell}} p_{a_{si}}) + (\sum_r a_r \pi_{a_r} p_{a_{rk}})(\sum_u a_u \pi_{a_{u\ell}} p_{a_{u\ell}}) \times$$

$$\times \left(\sum_i \sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{si}} q_{a_{si}} - \sum_{i \neq j} \sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{si}} p_{a_{sj}}\right)] \quad (5.31)$$

After the multiplication of the terms in the numerator of (5.31), the covariance of r_k and r_ℓ can be expressed as

$$\mathrm{cov}(r_k, r_\ell) =$$

$$= \left(\sum_s a_s \pi_{a_s} - \sum_i \sum_r a_r \pi_{a_r} p_{a_{ri}}\right)^{-2} \left(\sum_u a_u \pi_{a_u} - \sum_j \sum_v a_v \pi_{a_v} p_{a_{vj}}\right)^{-2} \times$$

$$\times [- \sum_s \sum_r \sum_u \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{sk}} p_{a_{s\ell}} +$$

$$+ \sum_i \sum_s \sum_r \sum_v \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_v \pi_{a_v}) p_{a_{sk}} p_{a_{s\ell}} p_{a_{vi}} +$$

$$+ \sum_j \sum_s \sum_r \sum_w \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_w \pi_{a_w}) p_{a_{sk}} p_{a_{s\ell}} p_{a_{wj}} -$$

$$- \sum_i \sum_j \sum_s \sum_v \sum_w \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_v \pi_{a_v})(a_w \pi_{a_w}) p_{a_{sk}} p_{a_{s\ell}} p_{a_{vi}} p_{a_{wj}} +$$

$$+ \sum_s \sum_r \sum_u \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{sk}} q_{a_{sk}} p_{a_{r\ell}} -$$

$$- \sum_i \sum_s \sum_r \sum_v \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_v \pi_{a_v}) p_{a_{sk}} q_{a_{sk}} p_{a_{r\ell}} p_{a_{vi}} -$$

$$- \sum_{j \neq k} \sum_{s} \sum_{r} \sum_{u} \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{sk}} p_{a_{r\ell}} p_{a_{sj}} +$$

$$+ \sum_{i} \sum_{j \neq k} \sum_{s} \sum_{r} \sum_{v} \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_v \pi_{a_v}) p_{a_{sk}} p_{a_{r\ell}} p_{a_{vi}} p_{a_{sj}} -$$

$$- \sum_{i} \sum_{s} \sum_{r} \sum_{u} \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{s\ell}} p_{a_{rk}} p_{a_{si}} +$$

$$+ \sum_{i} \sum_{j} \sum_{s} \sum_{r} \sum_{v} \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_v \pi_{a_v}) p_{a_{s\ell}} p_{a_{r\ell}} p_{a_{si}} p_{a_{vj}} +$$

$$+ \sum_{i} \sum_{s} \sum_{r} \sum_{u} \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{si}} q_{a_{si}} p_{a_{rk}} p_{a_{u\ell}} +$$

$$+ \sum_{i \neq j} \sum_{j} \sum_{s} \sum_{r} \sum_{v} \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_v \pi_{a_v}) p_{a_{si}} p_{a_{sj}} p_{a_{rk}} p_{a_{v\ell}}] .$$

(5.32)

After considerable simplification, expression (5.32) can be presented as

$\text{cov}(r_k, r_\ell) =$

$$(\sum_{s} a_s \pi_{a_s} - \sum_{i} \sum_{r} a_r \pi_{a_r} p_{a_{ri}})^{-2} (\sum_{s} a_s \pi_{a_s} - \sum_{j} \sum_{v} a_v \pi_{a_v} p_{a_{vj}})^{-2} \times$$

$$\times [\sum_{s} \sum_{r} \sum_{u} \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{sk}} (p_{a_r} - p_{a_{s\ell}}) +$$

$$+\sum_i\sum_s\sum_r\sum_u \frac{(a_s\pi_{a_s})^2}{n_{a_s}}(a_r\pi_{a_r})(a_u\pi_{a_u})\, p_{a_{s\ell}}(p_{a_{sk}}p_{a_{ri}}-p_{a_{rk}}p_{a_{si}})+$$

$$+\sum_j\sum_s\sum_r\sum_u \frac{(a_s\pi_{a_s})^2}{n_{a_s}}(a_r\pi_{a_r})(a_u\pi_{a_u})\, p_{a_{sk}}(p_{a_{s\ell}}p_{a_{rj}}-p_{a_{r\ell}}p_{a_{sj}})+$$

$$+\sum_i\sum_s\sum_r\sum_u \frac{(a_s\pi_{a_s})^2}{n_{a_s}}(a_r\pi_{a_r})(a_u\pi_{a_u})(p_{a_{si}}p_{a_{rk}}p_{a_{u\ell}}-p_{a_{sk}}p_{a_{r\ell}}p_{a_{ui}})$$

$$+\sum_i\sum_j\sum_s\sum_r\sum_u \frac{(a_s\pi_{a_s})^2}{n_{a_s}}(a_r\pi_{a_r})(a_u\pi_{a_u})\, p_{a_{s\ell}}\times$$

$$\times(p_{a_{sk}}p_{a_{vi}}p_{a_{rj}}-p_{a_{rk}}p_{a_{si}}p_{a_{vj}})+$$

$$+\sum_i\sum_j\sum_s\sum_r\sum_u \frac{(a_s\pi_{a_s})^2}{n_{a_s}}(a_r\pi_{a_r})(a_v\pi_{a_v})\, p_{a_{sj}}\times$$

$$\times(p_{a_{sk}}p_{a_{r\ell}}p_{a_{vi}}-p_{a_{rk}}p_{a_{v\ell}}p_{a_{si}})]. \qquad (5.33)$$

If M distinct value groups have the same life distribution, i.e., $p_{a_{si}} = p_{a_{ri}}$ for all s, r, and i, then it could be easily observed that $\mathrm{cov}(r_k, r_\ell) = 0$.

3. Derivation of large-sample variance of r_k.

The variance of retirement ratios within each interval can be computed by the following formula.

$$\mathrm{var}(r_k) = (\lambda - \sum_i \lambda_i)^{-4} [(\lambda - \sum_i \lambda_i)^2 \mathrm{var}(\varepsilon_k) +$$

$$+ 2\lambda_k(\lambda - \sum_i \lambda_i) \sum_i \mathrm{cov}(\varepsilon_k, \varepsilon_i) +$$

$$+ \lambda_k^2 (\sum_i \mathrm{var}(\varepsilon_i) + \sum_{i \neq j}\sum \mathrm{cov}(\varepsilon_i, \varepsilon_j))] \ . \qquad (5.34)$$

The covariance and variance of ε_i can be evaluated by (5.7) and (5.8), respectively. A substitution of λ, λ_i's, covariance and variance of ε_i into (5.34) permits the variance of r_k to be expressed as

$$\mathrm{var}(r_k) = (\sum_u a_u \pi_{a_u} - \sum_i \sum_v a_v \pi_{a_v} p_{a_{vi}})^{-4} \times$$

$$\times [(\sum_u a_u \pi_{a_u} - \sum_i \sum_r a_r \pi_{a_r} p_{a_{ri}})^2 (\sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{sk}} q_{a_{sk}}) -$$

$$- 2(\sum_r a_r \pi_{a_r} p_{a_{rk}})(\sum_u a_u \pi_{a_u} - \sum_j \sum_w a_w \pi_{a_w} p_{a_{wj}}) \times$$

$$\times (\sum_i \sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{sk}} p_{a_{si}}) + (\sum_r a_r \pi_{a_r} p_{a_{rk}})^2 \times$$

$$\times (\sum_i \sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{si}} q_{a_{si}} - \sum_{i \neq j} \sum \sum_s \frac{(a_s \pi_{a_s})^2}{n_{a_s}} p_{a_{si}} p_{a_{sj}})]$$

for $i, j = 1, 2, \ldots, k-1$

$s, r, u, v = 1, 2, \ldots, M.$ (5.35)

Multiplication of the terms in the numerator of (5.35) yields

$$\mathrm{var}(r_k) = (\sum_s a_s \pi_{a_s} - \sum_i \sum_s a_s \pi_{a_s} p_{a_{si}})^{-4} \times$$

$$\times [\sum_s \sum_u \sum_v \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_u \pi_{a_u})(a_v \pi_{a_v}) p_{a_{sk}} q_{a_{sk}} -$$

$$- 2 \sum_i \sum_s \sum_r \sum_u \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_u \pi_{a_u})(a_r \pi_{a_r}) p_{a_{sk}} q_{a_{sk}} p_{a_{ri}} +$$

$$+ \sum_i \sum_j \sum_s \sum_r \sum_w \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_w \pi_{a_w}) p_{a_{sk}} q_{a_{sk}} p_{a_{ri}} p_{a_{wj}} -$$

$$- 2 \sum_i \sum_s \sum_r \sum_u \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{sk}} p_{a_{rk}} p_{a_{si}} +$$

$$+ 2 \sum_i \sum_j \sum_s \sum_r \sum_w \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_w \pi_{a_w}) p_{a_{sk}} p_{a_{rk}} p_{a_{si}} p_{a_{wj}} +$$

$$+ \sum_i \sum_s \sum_r \sum_u \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{rk}} p_{a_{uk}} p_{a_{si}} q_{a_{si}} -$$

$$- \sum_{i \neq j} \sum_s \sum_r \sum_u \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{rk}} p_{a_{uk}} p_{a_{si}} p_{a_{sj}}].$$

(5.36)

After considerable simplification, the variance of r_k can be expressed as

$$\mathrm{var}(r_k) = (\sum_s a_s \pi_{a_s} - \sum_i \sum_s a_s \pi_{a_s} p_{a_{si}})^{-4} \times$$

$$\times [\sum_s \sum_u \sum_v n_{a_s}^{-1} (a_s \pi_{a_s})^2 (a_u \pi_{a_u})(a_v \pi_{a_v}) p_{a_{sk}} q_{a_{sk}} -$$

$$- 2 \sum_i \sum_s \sum_r \sum_u n_{a_s}^{-1} (a_s \pi_{a_s})^2 (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{sk}} p_{a_{ri}} +$$

$$+ \sum_i \sum_s \sum_r \sum_u n_{a_s}^{-1} (a_s \pi_{a_s})^2 (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_{rk}} p_{a_{uk}} p_{a_{si}} +$$

$$+ \sum_i \sum_j \sum_s \sum_r \sum_w n_{a_s}^{-1} (a_s \pi_{a_s})^2 (a_r \pi_{a_r})(a_w \pi_{a_w}) p_{a_{sk}} p_{a_{ri}} p_{a_{wj}} +$$

$$+ 2 \sum_i \sum_s \sum_r \sum_u n_{a_s}^{-1} (a_s \pi_{a_s})^2 (a_r \pi_{a_r})(a_u \pi_{a_u}) \times$$

$$\times p_{a_{sk}} (p_{a_{sk}} p_{a_{ri}} - p_{a_{rk}} p_{a_{si}}) +$$

$$+ \sum_i \sum_j \sum_s \sum_r \sum_w n_{a_s}^{-1} (a_s \pi_{a_s})^2 (a_r \pi_{a_r})(a_w \pi_{a_w}) \times$$

$$\times p_{a_{sk}} p_{a_{wj}} (p_{a_{rk}} p_{a_{si}} - p_{a_{sk}} p_{a_{ri}}) +$$

$$+ \sum_i \sum_j \sum_s \sum_r \sum_w n_{a_s}^{-1} (a_s \pi_{a_s})^2 (a_r \pi_{a_r})(a_w \pi_{a_w}) \times$$

$$\times p_{a_{rk}} p_{a_{si}} (p_{a_{sk}} p_{a_{wj}} - p_{a_{wk}} p_{a_{sj}})]. \qquad (5.37)$$

for $i, j = 1, 2, \ldots, k-1$;

$s, r, u, v, w = 1, 2, \ldots, M$.

Of special interest is the case in which all value-groups are subject to the same mortality characteristic,

$$p_{a_{si}} = p_{a_{ri}} \equiv p_i \text{ for all s, r and i.}$$

Then, (5.37) can be simplified to:

$$\mathrm{var}(r_k) = ((\sum_s a_s \pi_{a_s})(1 - \sum_i p_i))^{-4} \times$$

$$\times [\sum_s \sum_u \sum_v n_{a_s}^{-1}(a_s \pi_{a_s})^2 (a_u \pi_{a_u})(a_v \pi_{a_v})(p_k q_k - 2p_k \sum_i p_i +$$

$$+ p_k^2 \sum_i p_i + p_k(\sum_i p_i)^2] . \tag{5.38}$$

Equation (5.38) can be written in a compact form; that is

$$\mathrm{var}(r_k) = (\sum_{s=1}^{M} a_s \pi_{a_s})^{-2} (\sum_{s=1}^{M} n_{a_s}^{-1}(a_s \pi_{a_s})^2) \frac{(R_k - R_k^2)}{(1 - \sum_{i=1}^{k-1} p_i)} \tag{5.39}$$

where

$$R_k = \frac{p_k}{1 - \sum_{i=1}^{k-1} p_i} .$$

For M = 2 equation (5.39) gives

$$\mathrm{var}(r_k) = (a_1 \pi_{a_1} + a_2 \pi_{a_2})^{-2} (n_{a_1}^{-1}(a_1 \pi_{a_1})^2 + n_{a_2}^{-1}(a_2 \pi_{a_2})^2) \times$$

$$\times \frac{(R_k - R_k^2)}{(1 - \sum_{i=1}^{k-1} p_i)} .$$

The above equation is essentially the same as equation (5.25).

C. The Case of a Single Value Category

1. Expression of observed retirement ratios

The model for the single value category is essentially equivalent to model based on item counts. Again, the assumption is made that there is only a single vintage group which is composed of n large units.

The observed retirement ratios for each age interval can be derived from (5.1) by putting the restrictions

$a = 1$ and $b = 0$:

$$r_k = \frac{n_k}{n - \sum_{i=1}^{k-1} n_i} \; . \tag{5.40}$$

Note the index '1' is dropped since there is only one vintage group. Further, r_k can be approximated by the linear term expression:

$$r_k = \phi_k + \varepsilon_k \frac{\phi_k}{\lambda_k} + \phi_k \frac{\varepsilon^\circ{}_k}{(\lambda - \lambda^\circ{}_k)} \tag{5.41}$$

where

$$\lambda = 1; \; \lambda_k = p_k;$$

$$\phi_k = \frac{p_k}{(1 - \sum_i p_i)} \; ; \; \varepsilon_k = (\hat{p}_k - p_k).$$

2. Derivation of large-sample covariance of r_k and r_ℓ

As in the use of (5.16), estimates of covariance of retirement ratios can be computed by

$$\text{cov}(r_k, r_\ell) = (1 - \sum_i p_i)^{-2} (1 - \sum_j p_j)^{-2} \times$$

$$\times [-(1 - \sum_i p_i)(1 - \sum_j p_j) \frac{p_k\, p_\ell}{n} - p_k(1 - \sum_j p_j) \sum_i p_\ell\, p_i +$$

$$+ p_\ell(1 - \sum_i p_i) \frac{p_k\, q_k}{n} - p_\ell(1 - \sum_i p_i) \sum_{j \neq k} \frac{p_k\, p_j}{n} +$$

$$+ p_k\, p_\ell \sum_i \frac{p_i\, q_i}{n} - p_k\, p_\ell \sum_{i \neq j}\sum \frac{p_i\, p_j}{n}]. \tag{5.42}$$

After further simplification, (5.42) can be written as

$$\text{cov}(r_k, r_\ell) = n^{-1}(1 - \sum_i p_i)^{-2} (1 - \sum_j p_j)^{-2} \times$$

$$\times [-p_k\, p_\ell + p_k\, p_\ell \sum_i p_i + p_k\, p_\ell \sum p_j - p_k\, p_\ell \sum_i \sum_j p_i\, p_j -$$

$$- p_k\, p_\ell \sum_i p_i + p_k\, p_\ell \sum_i \sum_j p_i\, p_j + p_k\, p_\ell - p_k^2\, p_\ell -$$

$$- p_k\, p_\ell \sum_i p_i + p_k^2\, p_\ell \sum_i p_i + p_k\, p_\ell \sum_i \sum_j p_i\, p_j -$$

$$- p_k\, p_\ell \sum_j p_j - p_k^2\, p_\ell \sum_i p_i + p_k^2\, p_\ell + p_k\, p_\ell \sum_i p_i - p_k\, p_\ell \sum_i p_i^2 -$$

$$- p_k\, p_\ell \sum_i \sum_j p_i\, p_j + p_k\, p_\ell \sum_i p_i^2] = 0. \tag{5.43}$$

The above covariance of r_k and r_ℓ is literally zero. This is intuitively true since there is only one value group, hence, all units in the property die according to the same mortality characteristic.

3. Derivation of large-sample variance of r_k

The variance of retirement ratio in the k^{th} age interval can be evaluated as follows. As when formula (5.34) is used, the variance of r_k can be written as

$$\mathrm{var}(r_k) = \frac{\mathrm{var}(\hat{p}_k)}{(1 - \sum_i p_i)^2} + 2p_k \sum_i \frac{\mathrm{cov}(\hat{p}_k, \hat{p}_i)}{(1 - \sum_i p_i)^3} + \frac{p_k^2}{(1 - \sum p_i)^4} (\sum_i \mathrm{var}(\hat{p}_i) + \sum_{i \neq j}\sum \mathrm{cov}(\hat{p}_i, \hat{p}_j)) . \tag{5.44}$$

Variance and covariance of $\hat{p}$'s can be computed by (2.3) and (2.4), respectively.

A substitution of the variance-covariance of $\hat{p}$'s into (5.44), permits the variance of r_k to be presented by

$$\mathrm{var}(r_k) = \frac{1}{n} \left(\frac{p_k q_k}{(1 - \sum p_i)^2} - 2p_k \frac{\sum p_k p_i}{(1 - \sum p_i)^3} + \frac{p_k^2 \sum_i p_i q_i - p_k^2 \sum_{i \neq j}\sum p_i p_j}{(1 - \sum_i p_i)^4} \right) . \tag{5.45}$$

After further simplification, (5.45) can be expressed as

$$\text{var}(r_k) = \frac{1}{n}\left[\frac{p_k}{(1 - \sum_i p_i)^2} - \frac{p_k^2}{(1 - \sum_i p_i)^3}\right]$$

$$= \frac{[R_k - R_k^2]}{n(1 - \sum_i p_i)} \qquad (5.46)$$

where

R_k is defined as in (5.26).

The variance of retirement ratios may be computed by the other formula:

$$\text{var}(r_k) = \frac{1}{n}\left(\sum_{j=1}^{N} p_j\, d_j^2 - (\sum_j d_j\, p_j)^2\right) \qquad (5.47)$$

where

$$d_j = \frac{\partial r_k}{\partial p_j} \text{ for } j = 1, 2, .. N. \qquad (5.48)$$

For details of the derivation of (5.47), see Fleiss (1982).

Equation (5.40) in terms of $\hat{p}$'s may be written as

$$r_k = \frac{\hat{p}_k}{1 - \sum_{j=1}^{k-1} \hat{p}_j}\ .$$

r_k is then derived with respect to $\hat{p}_j$ for j = 1, 2, .. N:

$$\frac{\partial r_k}{\partial \hat{p}_k} = \frac{1}{(1 - \sum_j \hat{p}_j)}$$

$$\frac{\partial r_k}{\partial \hat{p}_j} = \frac{\hat{p}_k}{(1 - \sum_j \hat{p}_j)^2}\text{, for } j = 1, 2, \ldots, k-1 \tag{5.49}$$

$$\frac{\partial r_k}{\partial \hat{p}_j} = 0 \text{ for } j = k+1, k+2, \ldots, N.$$

$$\sum_{j=1}^{N} \hat{p}_j d_j = \sum_{j=1}^{k-1} \hat{p}_j \frac{\hat{p}_k}{(1 - \sum_j \hat{p}_j)^2} + \frac{\hat{p}_k}{(1 - \sum_j \hat{p}_j)}$$

$$= \hat{p}_k \left(\frac{1}{(1 - \sum_j \hat{p}_j)} + \frac{\sum_j \hat{p}_j}{(1 - \sum_j \hat{p}_j)^2} \right). \tag{5.50}$$

Upon the substitution of (5.49) and (5.50) into (5.47), the variance of r_k can be written as

$$\text{var}(r_k) = \frac{1}{n} \left[\frac{\hat{p}_k^2}{(1 - \sum_j \hat{p}_j)^4} \sum_j p_j + \frac{\hat{p}_k}{(1 - \sum_j \hat{p}_j)^2} - \hat{p}_k^2 \left(\frac{1}{(1 - \sum_j \hat{p}_j)} + \frac{\sum_j \hat{p}_j}{(1 - \sum_j \hat{p}_j)^2} \right)^2 \right]. \tag{5.51}$$

After further simplification, (5.47) may be written as

$$\text{var}(r_k) = \frac{1}{n} \left[\frac{\hat{p}_k}{(1 - \sum_j \hat{p}_j)} - \frac{\hat{p}_k^2}{(1 - \sum_j \hat{p}_j)^3} \right]. \tag{5.52}$$

The substitution $\hat{p}_j = p_j$ for all j into (5.52) gives

$$\mathrm{var}(r_k) = \frac{1}{n}\left(\frac{p_k}{(1 - \sum_j p_j)^2} - \frac{p_k^2}{(1 - \sum_j p_j)^3}\right) .$$

Thus,

$$\mathrm{var}(r_k) = \frac{(R_k - R_k^2)}{n(1 - \sum_{j=1}^{k-1} p_j)} \qquad (5.53)$$

where R_k is defined as in (5.26).

VI. A MODEL FOR THE JOINT DISCRETE DISTRIBUTION OF VINTAGE GROUP AND LIFE

The model discussed in Chapter V is essentially derived from the single vintage. The classification by value is statistically sound. However, the value-category may not exist in the accounting practices of an industrial firm. What usually is available is the classification by vintage.

Consider now property groups which are composed of several vintage groups. Needless to say each vintage group was installed during different years. It seems plausible that each vintage has a different life distribution. Management policy, economic conditions, inflation, technological breakthroughs, etc., all are responsible for each vintage having different mortality characteristic.

The assumption that all vintage groups have the same life distribution is usually made to simplify the analysis of data. It can be shown that under this assumption, the asymptotic covariances of the retirement ratios are zero, hence, weighted least square (or even least square) may be employed in fitting linear models to the retirement ratios.

This chapter presents the derivations of the retirement ratios and the corresponding estimates of the covariances and variances for industrial mortality data which depreciation engineers commonly use. It can be shown that with some modifications, the model in Chapter V remains applicable to these data.

Typical industrial mortality data can be cast by two-dimensional contingency table:

Table 1.

Year of placing	Size of vintage	Age at retirement 1	2	3	4	5	6	7	8	...
1	n_1	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}	n_{16}	n_{17}	n_{18}	...
2	n_2		n_{21}	n_{22}	n_{23}	n_{24}	n_{25}	n_{27}	n_{27}	...
3	n_3			n_{31}	n_{32}	n_{33}	n_{34}	n_{35}	n_{36}	...
4	n_4				n_{41}	n_{42}	n_{43}	n_{44}	n_{45}	...
5	n_5					n_{51}	n_{52}	n_{53}	n_{54}	...
6	n_6						n_{61}	n_{62}	n_{63}	...
7	n_7							n_{71}	n_{72}	...
8	n_8								n_{81}	...

The following definitions will be adopted:

n_{ij} denotes the number of item units from the i^{th} vintage retired during the age of interval j

$n_i = \sum_j n_{ij}$ represents the original number of item units from the i^{th} vintage group that are put in service at age zero

$\hat{p}_{ij}$ indicates the observed proportion of units from the i^{th} vintage retired during the j^{th} age interval

p_{ij} represents the true probability of a unit from the i^{th} vintage retired during the j^{th} age interval under whatever life distribution is assumed

$\pi_{ij} = \dfrac{n_i}{n_i + n_{i-1} + \cdots}$ is the proportion of units from the i^{th} vintage to the total units from all vintages which were included in the study

$$\varepsilon_{i_o j} = \sum_{i=i_o} \pi_{ij}(\hat{p}_{ij} - p_{ij})$$

$$\lambda_{i_o j} = \sum_{i=i_o} \pi_{ij}\, p_{ij}$$

$$\lambda = \sum_{j=1} \lambda_{i_o j} = \sum_i \sum_j \pi_{ij}\, p_{ij}$$

$$= \sum_i \pi_{ij}(\sum_j p_{ij}) = \sum_i \pi_{ij} = 1$$

e indicates the last (most recent) year that is included in the study of retirement experience

L denotes the width of the experience band[1] used in the study

w represents index of the width of experience band used in the study, w = 1, 2, ... L

A. Two-year Experience Band

1. Derivation of observed retirement ratios

To better understand the development of retirement ratios for mortality data from Table 1, consider a two-year experience band which begins with year six and ends at year seven. The placement band used in this case is years one through seven.

With the definition of retirement ratio,

---

[1]The calendar years for which the retirement experience of the total inventory (units or dollars retired or survives) is observed is called the experience or observation band.

The time period delimited by the year of installation is called the placement band.

r_1 = retirement ratio for the first year.

$$= \frac{\text{the number of units retired during the first age interval}}{\text{the number of units surviving at the beginning of age interval one}} \tag{6.1}$$

Notationally, (6.1) can be written as

$$r_1 = \frac{n_{61} + n_{71}}{n_6 + n_7}$$

$$= \frac{n_6}{n_6 + n_7} \cdot \frac{n_{61}}{n_6} + \frac{n_7}{n_6 + n_7} \cdot \frac{n_{71}}{n_7} \tag{6.2}$$

With the definitions of π's and $\hat{p}$'s, (6.2) can be expressed as

$$r_1 = \pi_{61}\, \hat{p}_{61} + \pi_{71}\, \hat{p}_{71}$$

$$= \pi_{61}(\hat{p}_{61} - p_{61}) + \pi_{71}(\hat{p}_{71} - p_{71})$$

$$+ \pi_{61}\, p_{61} + \pi_{71}\, p_{71} \tag{6.3}$$

In terms of λ_1 and ε_1, r_1 can be rewritten as

$$r_1 = \varepsilon_{61} + \lambda_{61} \tag{6.4}$$

Similarly, the retirement ratio in the second interval

$$r_2 = \frac{n_{52} + n_{62}}{n_5 + n_6 - (n_{51} + n_{61})} \tag{6.5}$$

When both numerator and denumerator of (6.5) are divided by $n_5 + n_6$, it gives

$$r_2 = \frac{\frac{n_5}{n_5 + n_6} \cdot \frac{n_{52}}{n_5} + \frac{n_5}{n_5 + n_6} \cdot \frac{n_{62}}{n_6}}{\frac{n_5 + n_6}{n_5 + n_6} - \left(\frac{n_5}{n_5 + n_6} \cdot \frac{n_{51}}{n_5} + \frac{n_6}{n_5 + n_6} \cdot \frac{n_{61}}{n_6}\right)} . \qquad (6.6)$$

Again, with the definitions of π's and $\hat{p}$'s, (6.6) can be written as

$$r_2 = \frac{\pi_{52}\, \hat{p}_{52} + \pi_{62}\, \hat{p}_{62}}{1 - (\pi_{52}\, \hat{p}_{51} + \pi_{62}\, \hat{p}_{61})} . \qquad (6.7)$$

In terms of ε's and λ's, (6.7) can be expressed as

$$r_2 = \frac{\lambda_{52} + \varepsilon_{52}}{1 - (\lambda_{51} + \varepsilon_{51})} . \qquad (6.8)$$

Similarly, r_3 can be expressed as

$$r_3 = \frac{\pi_{43}\, \hat{p}_{43} + \pi_{53}\, \hat{p}_{53}}{1 - (\pi_{43}\, \hat{p}_{41} + \pi_{53}\, \hat{p}_{51} + \pi_{43}\, \hat{p}_{42} + \pi_{53}\, \hat{p}_{52})} \qquad (6.9)$$

or

$$r_3 = \frac{\lambda_{43} + \varepsilon_{43}}{1 - (\lambda_{41} + \lambda_{42} + \varepsilon_{41} + \varepsilon_{42})} . \qquad (6.10)$$

It is important to note that λ's and ε's keep changing from one interval to the next.

The form of r's resembles the form of r_k in (5.3), hence, the linear Taylor approximation remains valid. However, the covariance formula is slightly changed due to the change in values of λ's and ε's.

2. Derivation of large-sample covariance of r_1 and r_2

The covariance of r_1 and r_2 can be derived as follows. As in (5.5), r_1 and r_2 can be expressed, respectively, as

$$r_1 = \lambda_{61} + \varepsilon_{61}$$

and

$$r_2 = \frac{\lambda_{52}}{(1 - \lambda_{51})} + \frac{\varepsilon_{52}}{(1 - \lambda_{51})} + \lambda_{52} \frac{\varepsilon_{51}}{(1 - \lambda_{51})^2} .$$

With the usual definition of covariance,

$$\mathrm{cov}(r_1, r_2) = \frac{1}{(1 - \lambda_{51})} \mathrm{cov}(\varepsilon_{52}, \varepsilon_{61}) + \frac{\lambda_{52}}{(1-\lambda_{51})^2} \mathrm{cov}(\varepsilon_{51}, \varepsilon_{61})$$

$$= \frac{1}{(1 - \lambda_{51})^2} [(1 - \lambda_{51}) \mathrm{cov}(\varepsilon_{52}, \varepsilon_{61})$$

$$+ \lambda_{52} \mathrm{cov}(\varepsilon_{51}, \varepsilon_{61})] . \tag{6.11}$$

According to Lemma 1 (Chiang, 1960a), the number of units retired from the i^{th} vintage group is distributed as multinomial with parameters n_i and p_{ij} for $j = 1, 2, \ldots, N$. Hence, it follows from (2.3) and (2.4) that

$$\mathrm{var}(\hat{p}_{ij}) = + \frac{p_{ij}\, q_{ij}}{n_i} \tag{6.12}$$

and

$$\mathrm{cov}(\hat{p}_{ij}, \hat{p}_{ij'}) = - \frac{p_{ij}\, p_{ij'}}{n_i} , \text{ for } j \neq j' . \tag{6.13}$$

The covariance terms in the square bracket of (6.11) are computed as follows:

$$\begin{aligned}\mathrm{cov}(\varepsilon_{52}, \varepsilon_{61}) &= \mathrm{cov}(\pi_{52}(\hat{p}_{52} - p_{52}) + \pi_{62}(\hat{p}_{62} - p_{62}),\\ &\qquad \pi_{61}(\hat{p}_{61} - p_{61}) + \pi_{71}(\hat{p}_{71} - p_{71}))\\ &= \mathrm{cov}(\pi_{61}(\hat{p}_{61} - p_{61}), \pi_{62}(\hat{p}_{62} - p_{62}))\\ &= \pi_{61} \ \pi_{62} \ \mathrm{cov}(\hat{p}_{61}, \hat{p}_{62})\end{aligned}$$

Equation (6.13) gives

$$\mathrm{cov}(\hat{p}_{61}, \hat{p}_{62}) = -\frac{p_{61} \ p_{62}}{n_6} .$$

Therefore,

$$\mathrm{cov}(\varepsilon_{52}, \varepsilon_{61}) = -\frac{\pi_{61} \ \pi_{62}}{n_6} p_{61} \ p_{62} . \qquad (6.14)$$

$$\begin{aligned}\mathrm{cov}(\varepsilon_{51}, \varepsilon_{61}) &= \mathrm{cov}(\pi_{52}(\hat{p}_{51} - p_{51}) + \pi_{62}(\hat{p}_{61} - p_{61}),\\ &\qquad \pi_{61}(\hat{p}_{61} - p_{61}) + \pi_{71}(\hat{p}_{71} - p_{71})).\\ &= \mathrm{cov}(\pi_{61}(\hat{p}_{61} - p_{61}), \pi_{62}(\hat{p}_{61} - p_{61}))\\ &= \pi_{61} \ \pi_{62} \ \mathrm{var}(\hat{p}_{61})\end{aligned}$$

Equation (6.12) gives

$$\mathrm{var}(\hat{p}_{61}) = \frac{p_{61}\, q_{61}}{n_6}$$

Thus,

$$\mathrm{cov}(\varepsilon_{51}, \varepsilon_{61}) = \frac{\pi_{61}\, \pi_{62}}{n_6}\, p_{61}\, q_{61}\ . \tag{6.15}$$

The substitution of (6.14) and (6.15) into (6.11) yields

$$\begin{aligned} \mathrm{cov}(r_1, r_2) &= (1 - \pi_{52}\, p_{51} - \pi_{62}\, p_{61})^{-1} \\ &\times [(1 - \pi_{52}\, p_{51} - \pi_{62}\, p_{61})(-\pi_{61}\, \pi_{62}\, \frac{p_{61}\, p_{62}}{n_6}) \\ &+ (\pi_{52}\, p_{52} + \pi_{62}\, p_{62})(\pi_{61}\, \pi_{62}\, \frac{p_{61}\, q_{61}}{n_6})]\ . \end{aligned} \tag{6.16}$$

After the multiplication of the terms in the numerator, (6.16) can be written as

$$\begin{aligned} \mathrm{cov}(r_1, r_2) &= (1 - \pi_{52}\, p_{51} - \pi_{62}\, p_{61})^{-1} \\ &\times [-\pi_{61}\, \pi_{62}\, \frac{p_{61}\, p_{62}}{n_6} + \pi_{52}\, \pi_{61}\, \pi_{62}\, \frac{p_{51}\, p_{61}\, p_{62}}{n_6} \\ &+ \pi_{61}\, \pi_{62}^2\, \frac{p_{61}^2\, p_{62}}{n_6} + \pi_{52}\, \pi_{61}\, \pi_{62}\, \frac{p_{61}\, q_{61}\, p_{52}}{n_6} \\ &+ \pi_{61}\, \pi_{62}^2\, \frac{p_{61}\, q_{61}\, p_{62}}{n_6}]\ . \end{aligned} \tag{6.17}$$

Under the condition that all vintage groups have the same life distribution,

$p_{ik} = p_k$, for all i and k:

$\text{cov}(r_1, r_2) =$

$$(1 - p_1)^{-1} \left[- \pi_{61} \pi_{62} \frac{p_1 p_2}{n_6} + \pi_{52} \pi_{61} \pi_{62} \frac{p_1^2 p_2}{n_6} + \pi_{61} \pi_{62}^2 \frac{p_1^2 p_2}{n_6} + \pi_{52} \pi_{61} \pi_{62} \frac{p_1 q_1 p_2}{n_6} + \pi_{61} \pi_{62}^2 \frac{p_1 q_1 p_2}{n_6}\right]. \tag{6.18}$$

With the substitution of $q_1 = 1 - p_1$ into (6.18) and after simplification, (6.18) can be expressed as

$$\text{cov}(r_1, r_2) = \frac{p_1 p_2}{n_6(1 - p_1)} [- \pi_{61} \pi_{62} + \pi_{52} \pi_{61} \pi_{62} + \pi_{61} \pi_{62}^2] \tag{6.19}$$

But

$$- \pi_{61} \pi_{62} + \pi_{61} \pi_{62}(\pi_{52} + \pi_{62}) =$$

$$- \pi_{61} \pi_{62} + \pi_{61} \pi_{62}(1) = 0 .$$

Hence, $\text{cov}(r_1, r_2) = 0$

B. Three-year Experience Band

1. Derivation of observed retirement ratios

Suppose now the band width of experience is extended to three years, using years seven, six and five.

Referring to contingency table (Table 1), the retirement ratio at the first age interval can be written as

$$r_1 = \frac{n_{51} + n_{61} + n_{71}}{n_5 + n_6 + n_7} \tag{6.20}$$

$$= \frac{n_5}{n_5 + n_6 + n_7} \frac{n_{51}}{n_5} + \frac{n_6}{n_5 + n_6 + n_7} \frac{n_{61}}{n_6} + \frac{n_7}{n_5 + n_6 + n_7} \frac{n_{71}}{n_7}$$

$$= \pi_{51} \hat{p}_{51} + \pi_{61} \hat{p}_{61} + \pi_{71} \hat{p}_{71}.$$

In terms of λ's and ε's, (6.20) can be represented by

$$r_1 = \lambda_{51} + \varepsilon_{51} \tag{6.21}$$

where

$$\lambda_{51} = \pi_{51} p_{51} + \pi_{61} p_{61} + \pi_{71} p_{71}$$

and

$$\varepsilon_{51} = \pi_{51}(\hat{p}_{51} - p_{51}) + \pi_{61}(\hat{p}_{61} - p_{61}) + \pi_{71}(\hat{p}_{71} - p_{71}) .$$

The retirement ratio for the second interval can be written as

$$r_2 = \frac{n_{42} + n_{52} + n_{62}}{n_4 + n_5 + n_6 - (n_{41} + n_{51} + n_{61})} . \tag{6.22}$$

If the numerator and denumerator of (6.22) are now divided by $(n_4 + n_5 + n_6)$ then

$$r_2 = \frac{\frac{n_4}{n_4 + n_5 + n_6} \cdot \frac{n_{42}}{n_4} + \frac{n_5}{n_4 + n_5 + n_6} \cdot \frac{n_{52}}{n_5} + \frac{n_6}{n_4 + n_5 + n_6} \cdot \frac{n_{62}}{n_6}}{1 - \left(\frac{n_4}{n_4 + n_5 + n_6} \cdot \frac{n_{41}}{n_4} + \frac{n_5}{n_4 + n_5 + n_6} \cdot \frac{n_{51}}{n_5} + \frac{n_6}{n_4 + n_5 + n_6} \cdot \frac{n_{61}}{n_6}\right)} \tag{6.23}$$

In terms of π's and $\hat{p}$'s, (6.23) can be expressed as

$$r_2 = \frac{\pi_{42}\ \hat{p}_{42} + \pi_{52}\ \hat{p}_{52} + \pi_{62}\ \hat{p}_{62}}{1 - (\pi_{42}\ \hat{p}_{41} + \pi_{52}\ \hat{p}_{51} + \pi_{62}\ \hat{p}_{61})} . \tag{6.24}$$

With the definition of ε's and λ's, (6.24) can be written as

$$r_2 = \frac{\varepsilon_{42} + \lambda_{42}}{1 - (\varepsilon_{41} + \lambda_{41})} . \tag{6.25}$$

The form of r_1 and r_2 remains the same as the form of r_k of (5.3). Therefore, the first order Taylor approximation to r_k's still holds. Variance and covariance formulas for r_k are slightly changed because of changes in values of ε's and λ's from one age interval to the next. This will become clear after evaluating covariance of r_1 and r_2 in the next section.

2. Derivation of large-sample covariance of r_1 and r_2

The basic formula for computing covariance of r_1 and r_2 is

$$\text{cov}(r_1, r_2) = \frac{1}{\lambda(\lambda - \lambda_1)^2} [(\lambda - \lambda_1) \text{cov}(\varepsilon_1, \varepsilon_2) + \lambda_2 \text{var}(\varepsilon_1)]. \tag{6.26}$$

Since λ_1 and ε_1 change in value from the first to second age intervals, formula (6.26) is also changed to become

$$\mathrm{cov}(r_1, r_2) = \frac{1}{(1 - \lambda_{41})^2} [(1 - \lambda_{41}) \mathrm{cov}(\varepsilon_{51}, \varepsilon_{42})$$

$$+ \lambda_{42} \mathrm{cov}(\varepsilon_{51}, \varepsilon_{41})]. \quad (6.27)$$

Covariance terms of ε in the square bracket in (6.27) are estimated as follows.

$$\mathrm{cov}(\varepsilon_{51}, \varepsilon_{42}) = \mathrm{cov}[\pi_{51}(\hat{p}_{51} - p_{51}) + \pi_{61}(\hat{p}_{61} - p_{61})$$

$$+ \pi_{71}(\hat{p}_{71} - p_{71}), \pi_{42}(\hat{p}_{42} - p_{42}) + \pi_{52}(\hat{p}_{52} - p_{52}) + \pi_{62}(\hat{p}_{62} - p_{62})]$$

$$= \mathrm{cov}(\pi_{51}(\hat{p}_{51} - p_{51}), \pi_{52}(\hat{p}_{52} - p_{52}))$$

$$+ \mathrm{cov}(\pi_{61}(\hat{p}_{61} - p_{61}, \pi_{62}(\hat{p}_{62} - p_{62}))$$

$$= \pi_{51} \pi_{52} \mathrm{cov}(\hat{p}_{51}, \hat{p}_{52}) + \pi_{61} \pi_{62} \mathrm{cov}(\hat{p}_{61}, \hat{p}_{62}). \quad (6.28)$$

But

$$\mathrm{cov}(\hat{p}_{51}, \hat{p}_{52}) = - \frac{p_{51} p_{52}}{n_5}$$

and

$$\mathrm{cov}(\hat{p}_{61}, \hat{p}_{62}) = - \frac{p_{61} p_{62}}{n_6}$$

Hence,

$$\mathrm{cov}(\varepsilon_{51}, \varepsilon_{42}) = - \frac{\pi_{51} \pi_{52}}{n_5} p_{51} p_{52} - \frac{\pi_{61} \pi_{62}}{n_6} p_{61} p_{62}. \quad (6.29)$$

$$\mathrm{cov}(\varepsilon_{51}, \varepsilon_{41}) = \mathrm{cov}[\pi_{51}(\hat{p}_{51} - p_{51}) + \pi_{61}(\hat{p}_{61} - p_{61}) + \pi_{71}(\hat{p}_{71} - p_{71}),$$

$$\pi_{42}(\hat{p}_{41} - p_{41}) + \pi_{52}(\hat{p}_{51} - p_{51}) + \pi_{62}(\hat{p}_{61} - p_{61})]$$

$$= \mathrm{cov}[\pi_{51}(\hat{p}_{51} - p_{51}), \pi_{52}(\hat{p}_{51} - p_{51})] +$$

$$\mathrm{cov}[\pi_{61}(\hat{p}_{61} - p_{61}), \pi_{62}(\hat{p}_{61} - p_{61})]$$

$$= \pi_{51}\ \pi_{52}\ \mathrm{var}(\hat{p}_{51}) + \pi_{61}\ \pi_{62}\ \mathrm{var}(p_{61})$$

But

$$\mathrm{var}(\hat{p}_{51}) = \frac{p_{51}\ q_{51}}{n_5}$$

and

$$\mathrm{var}(\hat{p}_{61}) = \frac{p_{61}\ q_{61}}{n_6}$$

Thus,

$$\mathrm{cov}(\varepsilon_{51}, \varepsilon_{41}) = + \pi_{51}\ \pi_{52} \frac{p_{51}\ q_{51}}{n_5} + \pi_{61}\ \pi_{62} \frac{q_{61}\ p_{61}}{n_6} . \qquad (6.30)$$

The substitution of λ's, (6.29) and (6.30) into equation (6.27) gives

$$\mathrm{cov}(r_1, r_2) = (1 - \pi_{42}\ p_{41} - \pi_{52}\ p_{51} - \pi_{62}\ p_{61})^{-1}$$

$$\times [(1 - \pi_{42}\ p_{41} - \pi_{52}\ p_{51} - \pi_{62}\ p_{61})$$

$$\times (- \pi_{51}\ \pi_{52} \frac{p_{51}\ p_{52}}{n_5} - \pi_{61}\ \pi_{62} \frac{p_{61}\ p_{62}}{n_6})$$

$$+ \ (\pi_{42}\ p_{42} + \pi_{52}\ p_{52} + \pi_{62}\ p_{62})(\pi_{51}\ p_{52}\ \frac{p_{51}\ q_{51}}{n_5} + \pi_{61}\ p_{62}\ \frac{p_{61}\ q_{61}}{n_6})] \tag{6.31}$$

The multiplication of the terms in the numerator of (6.31) yields:

$$\mathrm{cov}(r_1, r_2) = (1 - \pi_{42}\ \pi_{41} - \pi_{52}\ p_{51} - \pi_{62}\ p_{61})^{-1}$$

$$\times\ [-\ \pi_{51}\ \pi_{52}\ \frac{p_{51}\ p_{52}}{n_5} - \pi_{61}\ \pi_{62}\ \frac{p_{61}\ p_{62}}{n_6}$$

$$+\ \pi_{42}\ \pi_{51}\ \pi_{52}\ \frac{p_{41}\ p_{51}\ p_{52}}{n_5} + \pi_{42}\ \pi_{61}\ \pi_{62}\ \frac{p_{41}\ p_{61}\ p_{62}}{n_6}$$

$$+\ \pi_{52}^{\ 2}\ \pi_{51}\ \frac{p_{51}^{\ 2}\ p_{52}}{n_5} + \pi_{52}\ \pi_{61}\ \pi_{62}\ \frac{p_{51}\ p_{61}\ p_{62}}{n_6}$$

$$+\ \pi_{51}\ \pi_{51}\ \pi_{62}\ \frac{p_{51}\ p_{52}\ p_{61}}{n_5} + \pi_{61}\ \pi_{62}^{\ 2}\ \frac{p_{61}^{\ 2}\ p_{62}}{n_6}$$

$$+\ \pi_{42}\ \pi_{51}\ \pi_{52}\ \frac{p_{42}\ p_{51}\ q_{51}}{n_5} + \pi_{42}\ \pi_{61}\ \pi_{62}\ \frac{p_{42}\ p_{61}\ q_{61}}{n_6}$$

$$+\ \pi_{51}\ \pi_{52}^{\ 2}\ \frac{p_{51}\ q_{51}\ p_{52}}{n_5} + \pi_{52}\ \pi_{61}\ \pi_{62}\ \frac{p_{52}\ p_{61}\ q_{61}}{n_6}$$

$$+\ \pi_{51}\ \pi_{52}\ \pi_{62}\ \frac{p_{51}\ q_{51}\ p_{62}}{n_5} + \pi_{61}\ \pi_{62}^{\ 2}\ \frac{p_{61}\ q_{61}\ p_{62}}{n_6}\ ]. \tag{6.32}$$

Under the condition that all vintage groups are subject to the same mortality characteristic,

$p_{ik} \equiv p_k$ for all possible values of i and k,

$$cov(r_1, r_2) - (1 - \pi_{42}\, p_1 - \pi_{52}\, p_1 - \pi_{62}\, p_1)^{-1}$$

$$\times [- \pi_{51}\, \pi_{52} \frac{p_1\, p_2}{n_5} - \pi_{61}\, \pi_{62} \frac{p_1\, p_2}{n_6}$$

$$+ \pi_{42}\, \pi_{51}\, \pi_{52} \frac{p_1^2\, p_2}{n_5} + \pi_{42}\, \pi_{61}\, \pi_{62} \frac{p_1^2\, p_2}{n_6}$$

$$+ \pi_{52}^2\, \pi_{51} \frac{p_1^2\, p_2}{n_5} + \pi_{52}\, \pi_{61}\, \pi_{62} \frac{p_1^2\, p_2}{n_6}$$

$$+ \pi_{51}\, \pi_{52}\, \pi_{62} \frac{p_1^2\, p_2}{n_5} + \pi_{61}\, \pi_{62}^2 \frac{p_1^2\, p_2}{n_6}$$

$$+ \pi_{42}\, \pi_{51}\, \pi_{52} \frac{p_1\, q_1\, p_2}{n_5} + \pi_{42}\, \pi_{61}\, \pi_{62} \frac{p_1\, q_1\, p_2}{n_6}$$

$$+ \pi_{51}\, \pi_{52}^2 \frac{p_1\, q_1\, p_2}{n_5} + \pi_{52}\, \pi_{61}\, \pi_{62} \frac{p_1\, q_1\, p_2}{n_6}$$

$$+ \pi_{51}\, \pi_{52}\, \pi_{62} \frac{p_1\, q_1\, p_2}{n_5} + \pi_{61}\, \pi_{62}^2 \frac{p_1\, q_1\, p_2}{n_6}]. \qquad (6.33)$$

With the substitution of $q_1 = 1 - p_1$ into (6.33) and after further simplification, (6.33) can be written as

$$cov(r_1, r_2) =$$

$$\frac{p_1 p_2}{(1 - p_1)} [n_5^{-1}(- \pi_{51}\, \pi_{52} + \pi_{42}\, \pi_{51}\, \pi_{52} + \pi_{51}\, \pi_{52}^2 + \pi_{51}\, \pi_{52}\, \pi_{62})$$

$$+ n_6^{-1}(- \pi_{61}\, \pi_{62} + \pi_{42}\, \pi_{61}\, \pi_{62} + \pi_{52}\, \pi_{61}\, \pi_{62} + \pi_{61}\, \pi_{62}^2)] .$$

But

$$-\pi_{51}\,\pi_{52} + \pi_{51}\,\pi_{52}(\pi_{42} + \pi_{52} + \pi_{62})$$

$$= -\pi_{51}\,\pi_{52} + \pi_{51}\,\pi_{52} \cdot 1 = 0$$

Similarly,

$$-\pi_{61}\,\pi_{62} + \pi_{61}\,\pi_{62}(\pi_{42} + \pi_{52} + \pi_{62})$$

$$= -\pi_{61}\,\pi_{6}\;\; + \pi_{61}\,\pi_{62} \cdot 1 = 0$$

Hence,

$$\mathrm{cov}(r_1,\ r_2) = 0$$

Thus, for the three-year experience band covariance of r_1 and r_2 remains asymptotically zero.

C. L-year Experience Band Based on Item Counts

1. Derivation of observed retirement ratios

Analogously, for the case of data aggregated over vintage groups, where (e) represents the most recent vintage year, and (L) represents the experience band used in the study, the retirement ratios for the k^{th} age interval can be written as

$$r_k = \frac{\sum_{w=1}^{L} n_{e-w-k+2,k}}{\sum_{w=1}^{L} n_{e-w-k+2} - \sum_{j=1}^{k-1} \sum_{w=1}^{L} n_{e-w-k+2,j}} \qquad (6.34)$$

When the original data recorded in terms of units of dollars, r_k may be further represented as

$$r_k = \frac{\sum_{w=1}^{L} a_{e-w-k+2}\, n_{e-w-k+2,k}}{\sum_{w=1}^{L} a_{e-w-k+2}\, n_{e-w-k+2} - \sum_{i=1}^{k-1} \sum_{w=1}^{L} a_{e-w-k+2}\, n_{e-w-k+2,i}} \tag{6.35}$$

It is worth noting here that $n_{e-w-k+2,k}$ and $a_{e-w-k+2}$ may not be available. Only the lump sums in dollars may be known. For the purpose of this study assume that each vintage size, $n_{e-w-k+2}$, is known. To illustrate how formula (6.34) works, suppose the most recent vintage year, $e = 7$, and let $L = 2$ and $k = 3$. Then,

$$r_3 = \frac{\sum_{w=1}^{2} n_{7-w-3+2,3}}{\sum_{w=1}^{2} n_{7-w-3+2} - \sum_{i=1}^{2} \sum_{w=1}^{2} n_{7-w-3+2,j}} \tag{6.36}$$

If each term in the numerator and denumerator of (6.36) is written out, then

$$r_3 = \frac{n_{53} + n_{43}}{n_5 + n_4 - (n_{51} + n_{52} + n_{41} + n_{42})} .$$

Both numerator and denumerator of (6.34) are now divided by $\sum_{w=1}^{L} n_{e-w-k+2}$ which gives

$$r_k = \frac{\sum_w \frac{n_{e-w-k+2}}{(\sum_w n_{e-w-k+2})} \frac{n_{e-w-k+2,k}}{n_{e-w-k+2}}}{1 - \sum_i \sum_w \frac{n_{e-w-k+2}}{(\sum_w n_{e-w-k+2})} \frac{n_{e-w-k+2,i}}{n_{e-w-k+2}}} \tag{6.37}$$

If the definitions of π's and $\hat{p}$'s are employed, then (6.37) can be written as

$$r_k = \frac{\sum_w \pi_{e-w-k+2,k} \hat{p}_{e-w-k+2,k}}{1 - \sum_i \sum_w \pi_{e-w-k+2,k} \hat{p}_{e-w-k+2,i}} . \tag{6.38}$$

In terms of λ's and ε's, (6.38) may be represented as

$$r_k = \frac{\lambda_{e-L-k+2,k} + \varepsilon_{e-L-k+2,k}}{1 - \sum_i \lambda_{e-L-k+2,i} - \sum_i \varepsilon_{e-L-k+2,i}} , \tag{6.39}$$

where

$$\lambda_{e-L-k+2,i} = \sum_w \pi_{e-w-k+2,k} p_{e-w-k+2,i}$$

$$\varepsilon_{e-L-k+2,i} = \sum_w \pi_{e-w-k+2,k} (\hat{p}_{e-w-k+2,i} - p_{e-w-k+2,i})$$

for $i = 1, 2, \ldots, k-1$;

$w = 1, 2, \ldots, L$.

As before, r_k can be approximated by linear order terms of the Taylor expansion:

$$r_k = \phi_{e-L-k+2,k} + \frac{\varepsilon_{e-L-k+2,k}\ \phi_{e-L-k+2,k}}{\lambda_{e-L-k+2,k}}$$

$$+ \varepsilon^\circ_{e-L-k+2,k} \frac{\phi_{e-L-k+2,k}}{(1 - \lambda^\circ_{e-L-k+2,k})} \qquad (6.40)$$

where

$$\lambda^\circ_{e-L-k+2,k} = \sum_{i=1}^{k-1} \lambda_{e-L-k+2,i}$$

$$\varepsilon^\circ_{e-L-k+2,k} = \sum_{i=1}^{k-1} \varepsilon_{e-L-k+2,i}$$

and

$$\phi_{e-L-k+2,k} = \frac{\lambda_{e-L-k+2,k}}{(1 - \lambda^\circ_{e-L-k+2,k})}$$

2. Derivation of large-sample covariance of r_1 and r_2

Asymptotic covariance of retirement ratios in the first and second intervals can be derived as follows.

For k = 1 and k = 2, (6.40) gives

$$r_1 = \lambda_{e-L+1,1} + \varepsilon_{e-L+1,1}$$

and

$$r_2 = \frac{\lambda_{e-L,2}}{(1 - \lambda_{e-L,1})} + \frac{\varepsilon_{e-L,2}}{(1 - \lambda_{e-L,1})} + \lambda_{e-L,2} \frac{\varepsilon_{e-L,1}}{(1 - \lambda_{e-L,1})^2} . \qquad (6.41)$$

By the definition of covariance:

$$\mathrm{cov}(r_1, r_2) = (1 - \lambda_{e-L,1})^{-2}$$

$$\times [(1 - \lambda_{e-L,1}) \,\mathrm{cov}(\varepsilon_{e-L+1,1}, \varepsilon_{e-L,2})$$

$$+ \lambda_{e-L,2} \,\mathrm{cov}(\varepsilon_{e-L,1}, \varepsilon_{e-L+1,1})] \qquad (6.42)$$

Before the computation of the covariances of ε's in (6.42) a few useful formulas are derived:

$$\mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,j})$$

$$= \mathrm{cov}[\sum_u \pi_{e-u-k+2,k}(\hat{p}_{e-u-k+2,i} - p_{e-u-k+2,i}),$$

$$\sum_w \pi_{e-w-\ell+2,\ell}(\hat{p}_{e-w-\ell+2,j} - p_{e-w-\ell+2,j})]$$

$$= \sum_w \mathrm{cov}(\pi_{e-w-\ell+2,k}(\hat{p}_{e-w-\ell+2,i} - p_{e-w-\ell+2,i}),$$

$$\pi_{e-w-\ell+2,\ell}(\hat{p}_{e-w-\ell+2,j} - p_{e-w-\ell+2,j}))$$

$$+ \sum_w \sum_{u \neq w+\ell-k} \mathrm{cov}(\pi_{e-u-k+2,k}(\hat{p}_{e-u-k+2,i} - p_{e-u-k+2,i}),$$

$$\pi_{e-w-\ell+2,\ell}(\hat{p}_{e-w-\ell+2,j} - p_{e-w-\ell+2,j}))$$

$$= -\sum_{w=1}^{L+k-\ell} n^{-1}_{e-w-\ell+2}\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}\ p_{e-w-\ell+2,j} + 0$$

$= 0$ for $j - i \geq L$.

Thus,

$$\text{cov}(\varepsilon_{e-L-k+2,i},\ \varepsilon_{e-L-\ell+2,j})$$

$$= -\sum_{w=1}^{L+k-\ell} n^{-1}_{e-w-\ell+2}\ \pi_{e-w-\ell+w,k}\ \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,i}\ p_{e-w-\ell+2,j} \qquad (6.43)$$

It is important to note here that the covariance of terms which come from different vintage groups are zero. For $i = j$ (6.43) becomes:

$$\text{cov}(\varepsilon_{e-L-k+2,i},\ \varepsilon_{e-L-\ell+2,i})$$

$$= \sum_{w=1}^{L+k-\ell} n^{-1}_{e-w-\ell+2}\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-w-\ell+2,i}\ q_{e-w-\ell+2,i} \qquad (6.44)$$

It is worth noting here that

$$\pi_{e-w-\ell+2,k} \neq \pi_{e-w-\ell+2,\ell} \text{ for } k \neq \ell$$

and

$$\sum_{w=1}^{L} \pi_{e-w-i+2,i} = 1 \text{ for all } i.$$

Now, for $k = 1$, $\ell = 2$, $i = 1$ and $j = 2$, formula (6.43) provides:

$$\text{cov}(\varepsilon_{e-L+1,1},\ \varepsilon_{e-L,2}) = -\sum_{w=1}^{L-1} n^{-1}_{e-w}\ \pi_{e-w,1}\ \pi_{e-w,2}\ p_{e-w,1}\ p_{e-w,2} \qquad (6.45)$$

and for $k = 1$, $\ell = 2$, $i = j = 1$, (6.44) gives

$$\mathrm{cov}(\varepsilon_{e-L+1,1}, \varepsilon_{e-L,1}) = \sum_{w=1}^{L-1} n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} \; p_{e-w,1} \, q_{e-w,1} \tag{6.46}$$

Upon substitution of $\lambda_{e-w,1}$, $\lambda_{e-w,2}$, (6.45) and (6.46) into (6.42),

$$\mathrm{cov}(r_1, r_2) = (1 - \sum_w \pi_{e-w,1} \, p_{e-w,1})^{-2}$$

$$[(1 - \sum_w \pi_{e-w,1} \, p_{e-w,1})(- \sum_w n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} \, p_{e-w,1} \, p_{e-w,2})$$

$$+ (\sum_w \pi_{e-w,2} \, p_{e-w,2})(\sum_w n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} \, p_{e-w,1} \, q_{e-w,1})] \tag{6.47}$$

After the multiplication of the terms in the numerator, (6.47) can be written as:

$$\mathrm{cov}(r_1, r_2) = (1 - \sum_w \pi_{e-w,1} \, p_{e-w,1})^{-2}$$

$$\times [- \sum_w n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} \, p_{e-w,1} \, p_{e-w,2}$$

$$+ \sum_u \sum_w n_{e-w}^{-1} \pi_{e-u,1} \pi_{e-w,1} \pi_{e-w,2} \, p_{e-u,1} \, p_{e-u,1} \, p_{e-w,2}$$

$$+ \sum_u \sum_w n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} \pi_{e-u,2} \, p_{e-w,1} \, q_{e-w,1} \, p_{e-u,2}] . \tag{6.48}$$

If it is assumed that all vintage groups die according to the same mortality characteristic, i.e.,

$$p_{e-w,k} \equiv p_k \text{ for all } w \text{ and } k$$

then $\mathrm{cov}(r_1, r_2)$ is further simplified:

$$\mathrm{cov}(r_1, r_2) = (1 - p_1)^{-2}$$

$$\times [- \sum_w n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} p_1 p_2$$

$$+ \sum_u \sum_w n_{e-w}^{-1} \pi_{e-u,1} \pi_{e-w,1} \pi_{e-w,2} p_1^2 p_2$$

$$+ \sum_u \sum_w n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} \pi_{e-u,w} p_1(1 - p_1) p_2]$$

Hence,

$$\mathrm{cov}(r_1, r_2) \simeq$$

$$= (1 - p_1)^{-2} [- \sum_w n_{e-w}^{-1} \pi_{e-w,1} \pi_{e-w,2} p_1 p_2 + \sum_u \sum_w n_{e-w}^{-1} \pi_{e-w,1}$$

$$\times \pi_{e-w,2} \pi_{e-u,2} p_1 p_2]$$

But,

$$\sum_{u=1}^{L} \pi_{e-u,2} = 1.$$

Therefore,

$$\mathrm{cov}(r_1, r_2) = p_1\, p_2 (1 - p_1)^{-2}$$

$$\times \left(\sum_w n_{e-w}^{-1}\, \pi_{e-w,1}\, \pi_{e-w,2} - \sum_w n_{e-w}^{-1}\, \pi_{e-w,1}\, \pi_{e-w,2}\right) = 0\ .$$

3. Derivation of large-sample covariance of r_k and r_ℓ

Asymptotic covariance of retirement ratios for the age intervals k and ℓ, where $k < \ell$ can be derived as follows.

For k = k and k = ℓ equation (6.40) respectively gives

$$r_k = \frac{\lambda_{e-L-k+2,k}}{(1 - \sum_i \lambda_{e-L-k+2,i})} + \frac{\varepsilon_{e-L-k+2,k}}{(1 - \sum_i \lambda_{e-L-k+2,i})}$$

$$+ \frac{\lambda_{e-L-k+2,k}\, (\sum_i \varepsilon_{e-L-k+2,i})}{(1 - \sum_i \lambda_{e-L-k+2,i})^2}$$

and

$$r_\ell = \frac{\lambda_{e-L-\ell+2,\ell}}{(1 - \sum_j \lambda_{e-L-\ell+2,j})} + \frac{\varepsilon_{e-L-\ell+2,\ell}}{(1 - \sum_j \lambda_{e-L-\ell+2,j})}$$

$$+ \frac{\lambda_{e-L-\ell+2,\ell}\, (\sum_j \varepsilon_{e-L-\ell+2,j})}{(1 - \sum_j \lambda_{e-L-\ell+2,j})^2}$$

From the definition of covariance:

$$\mathrm{cov}(r_k, r_\ell) = (1 - \sum_i \lambda_{e-L-k+2,i})^{-2} (1 - \sum_j \lambda_{e-L-\ell+2,j})^{-2}$$

$$\times [(1 - \sum_i \lambda_{e-L-k+2,i})(1 - \sum_j \lambda_{e-L- +2,j}) \mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,\ell})$$

$$+ \lambda_{e-L-\ell+2,\ell}(1 - \sum_i \lambda_{e-L-k+2,i})(\mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,k})$$

$$+ \sum_{j \neq k} \mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,j}))$$

$$+ \lambda_{e-L-k+2,k}(1 - \sum_j \lambda_{e-L-\ell+2,j}) \sum_i \mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,\ell})$$

$$+ \lambda_{e-L-k+2,k} \lambda_{e-L-\ell+2,\ell} (\sum_i \mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,i})$$

$$+ \sum_{i \neq j} \sum \mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,j}))] \qquad (6.49)$$

for $i = 1, 2, \ldots, k-1$

$j = 1, 2, \ldots, \ell-1$

All the covariance-terms in the square bracket of (6.49) are computed by formulas (6.43) and (6.44):

$$\mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,\ell})$$

$$= - \sum_w n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k} p_{e-w-\ell+2,\ell} \qquad (6.50)$$

$$\mathrm{cov}(\varepsilon_{e-L-k+2,k}, \sum_j \varepsilon_{e-L-\ell+2,j})$$

$$= \mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-k+2,k}) +$$

$$+ \mathrm{cov}(\varepsilon_{e-L-k+2,k}, \sum_{j \neq k} \varepsilon_{e-L-\ell+2,j})$$

$$= \sum_w n^{-1}_{e-w-k+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k} q_{e-w-\ell+2,k}$$

$$- \sum_{j \neq k} \sum n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k} p_{e-w-\ell+2,j} \qquad (6.51)$$

$$\mathrm{cov}(\varepsilon_{e-L-\ell+2,\ell} \sum_i \varepsilon_{e-L-k+2,i})$$

$$= \sum_i \mathrm{cov}(\varepsilon_{e-w-\ell+2,\ell}, \varepsilon_{e-w-k+2,i})$$

$$= - \sum_i \sum_w n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,i} p_{e-w-\ell+2,\ell} \qquad (6.52)$$

$$\mathrm{cov}(\sum_i \varepsilon_{e-L-k+2,i}, \sum_j \pi_{e-L-\ell+2,j})$$

$$= \sum_i \mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,i})$$

$$+ \sum_{i \neq j} \sum \mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,j})$$

$$= \sum_i \sum_w n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,i} q_{e-w-\ell+2,i}$$

$$- \sum_{\substack{i \\ i \neq j}} \sum_j \sum_w n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,i} p_{e-w-\ell+2,j} . \quad (6.53)$$

Upon the substitution of terms (6.50), (6.51), (6.52), (6.53) and λ's into equation (6.49), covariance between retirement ratios can be written as

$$\mathrm{cov}(r_k, r_\ell) =$$

$$(1 - \sum_i \sum_u \pi_{e-u-k+2,k} p_{e-u-k+2,i})^{-2} (1 - \sum_j \sum_v \pi_{e-v-\ell+2,\ell} p_{e-v-\ell+2,j})^{-2}$$

$$\times [(1 - \sum_i \sum_u \pi_{e-u-k+2,k} p_{e-u-k+2,i})(1 - \sum_j \sum_v \pi_{e-v-\ell+2,\ell} p_{e-v-\ell+2,j})$$

$$\times (- \sum_w n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k} p_{e-w-\ell+2,\ell})$$

$$+ (\sum_v \pi_{e-v-\ell+2,\ell} p_{e-v-\ell+2,\ell})(1 - \sum_i \sum_u \pi_{e-u-k+2,k} p_{e-w-k+2,i})$$

$$\times (\sum_w n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k} q_{e-w-\ell+2,k}$$

$$- \sum_{j \neq k} \sum_w n^{-1}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k} p_{e-w-\ell+2,j})$$

$$+ (\sum_u \pi_{e-u-k+2,k} p_{e-u-k+2,k})(1 - \sum_j \sum_v \pi_{e-v-\ell+2,\ell} p_{e-v-\ell+2,j})$$

$$\times \Big(- \sum_i \sum_w n^{-1}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}\, p_{e-w-\ell+2,\ell}\Big)$$

$$+ \Big(\sum_u \pi_{e-u-k+2,k}\, p_{e-u-k+2,k}\Big)\Big(\sum_v \pi_{e-v-\ell+2,\ell}\, p_{e-v-\ell+2,\ell}\Big)$$

$$\times \Big(\sum_i \sum_w n^{-1}_{e-w-\ell+w}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}\, q_{e-w-\ell+2,i}$$

$$- \sum_{\substack{i\\ i\neq j}} \sum_j \sum_w n^{-1}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}\, p_{e-w-\ell+2,j}\Big)\Big] \qquad (6.54)$$

The multiplication of the terms in the numerators of (6.54) yields:

$$\mathrm{cov}(r_k,\ r_\ell) =$$

$$\Big(1 - \sum_i \sum_u \pi_{e-u-k+2,k}\, p_{e-u-k+2,i}\Big)^{-2} \Big(1 - \sum_j \sum_v \pi_{e-v-\ell+2,\ell}\, p_{e-v-\ell+2,j}\Big)^{-2}$$

$$\times \Big[- \sum_w n^{-1}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}\, p_{e-w-\ell+2,\ell}$$

$$+ \sum_{iuw}\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-u-k+2,i}$$

$$\times\, p_{e-w-\ell+2,k}\, p_{e-w-\ell+2,\ell}$$

$$+ \sum_{jvw}\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-v-\ell+2,j}$$

$$\times\, p_{e-w-\ell+2,k}\, p_{e-w-\ell+2,\ell}$$

$$- \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times\, p_{e-u-k+2,i}\, p_{e-v-\ell+2,j}\, p_{e-w-k+2,k}\, p_{e-w-\ell+2,\ell}$$

$$+ \sum_{vw}\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-v-\ell+2,\ell}\, p_{e-w-\ell+2,k}$$

$$\times\, q_{e-w-\ell+2,k}$$

$$- \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times\, p_{e-v-\ell+2,\ell}\, p_{e-u-k+2,i}\, p_{e-w-\ell+2,k}\, q_{e-w-\ell+2,k}$$

$$- \sum_{j\neq k} \sum_{v} \sum_{w} n^{-1}_{e-w-\ell+2}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}$$

$$\times\, p_{e-v-\ell+2,\ell}\, p_{e-w-\ell+2,j}$$

$$+ \sum_{i} \sum_{\substack{j \\ j\neq k}} \sum_{u} \sum_{v} \sum_{w} n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times\, p_{e-v-\ell+2,\ell}\, p_{e-u-k+w,i}\, p_{e-w-\ell+2,k}\, p_{e-w-\ell+2,j}$$

$$- \sum_{iuw}\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-u-k+2,k}$$

$$\times\, p_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times\, p_{e-u-k+2,k}\, p_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}\, p_{e-v-\ell+2,j}$$

$$+ \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times\, p_{e-u-k+2,k}\, p_{e-v-\ell+2,\ell}\, p_{e-w-\ell+2,i}\, q_{e-w-\ell+2,i}$$

$$- \sum_{i\neq j}\sum_{u}\sum_{v}\sum_{w} n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times\, p_{e-u-k+2,k}\, p_{e-v-\ell+2,\ell}\, p_{e-w-\ell+2,i}\, p_{e-w-\ell+2,j}] . \tag{6.55}$$

for $i = 1, 2, \ldots, k-1$

$j = 1, 2, \ldots, \ell-1$

$w = 1, 2, \ldots, L+k-\ell;\ u, v = 1, 2, \ldots, L$

After further simplification (6.55) may be presented as

$\mathrm{cov}(r_k, r_\ell) =$

$$(1 - \sum_{i}\sum_{u} \pi_{e-u-k+2,k}\, p_{e-u-k+2,i})^{-2}\, (1 - \sum_{i}\sum_{v} \pi_{e-v-\ell+2,\ell}\, p_{e-v-\ell+2,j})^{-2}$$

$$\times\, [\sum_{v}\sum_{w} n^{-1}_{e-w-\ell+2}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}$$

$$\times (p_{e-v-\ell+2,\ell} - p_{e-w-\ell+2,\ell})$$

$$+ \sum_{iuw}\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,\ell}$$

$$\times (p_{e-w-\ell+2,k}\, p_{e-u-k+2,i} - p_{e-u-k+2,k}\, p_{e-w-\ell+2,i})$$

$$+ \sum_{juw}\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times p_{e-w-\ell+2,k}(p_{e-w-\ell+2,\ell}\, p_{e-u-\ell+2,j} - p_{e-u-\ell+2,\ell}\, p_{e-w-\ell+2,j})$$

$$+ \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times p_{e-v-\ell+2,\ell}(p_{e-u-k+2,k}\, p_{e-w-\ell+2,i} - p_{e-u-k+2,i}\, p_{e-w-\ell+2,k})$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-\ell+2,\ell}\, \pi_{e-v-k+2,k}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times p_{e-u-\ell+2,\ell}\, p_{e-w-\ell+2,j}(p_{e-v-k+2,i}\, p_{e-w-\ell+2,k} - p_{e-v-k+2,k}$$

$$\times p_{e-w-\ell+2,i})$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}$$

$$\times p_{e-v-\ell+2,j}(p_{e-u-k+2,k}\, p_{e-w-\ell+2,i} - p_{e-u-k+2,i}\, p_{e-w-k+2,k})]. \qquad (6.56)$$

Of special interest here is the case in which all vintage groups die according to the same mortality characteristic,

$$p_{e-w-k+2,j} = p_{e-u-k+2,j} \equiv p_j \text{ for all u, v, and j.}$$

It can be directly observed that $\text{cov}(r_k, r_\ell) = 0$.

4. Derivation of large-sample variance of r_k

The variance of r_k can be estimated by the following formula.

$$\begin{aligned}
\text{var}(r_k) &= (1 - \sum_i \lambda_{e-L-k+2,i})^{-4} \\
&\times [(1 - \sum_i \lambda_{e-L-k+2,i})^2 \text{var}(\varepsilon_{e-L-k+2,k}) \\
&+ 2\lambda_{e-L-k+2,k} (1 - \sum_i \lambda_{e-L-k+2,i}) \\
&\times (\sum_i \text{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-k+2,i})) \\
&+ \lambda^2_{e-L-k+2,k}(\sum_i \text{var}(\varepsilon_{e-L-k+2,i}) \\
&+ \sum_{i \neq j}\sum \text{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-k+2,j}))]
\end{aligned} \tag{6.57}$$

Covariance and variance terms of (6.57) can be estimated by using formulas (6.43) and (6.44):

$$\begin{aligned}
\text{var}(\varepsilon_{e-L-k+2,k}) &= \text{var}(\sum_w \pi_{e-w-k+2,k} (\hat{p}_{e-w-k+2,k} - p_{e-w-k+2,k})) \\
&= \sum_w n^{-1}_{e-w-k+2} \pi^2_{e-w-k+2,k} p_{e-w-k+2,k} q_{e-w-k+2,k}
\end{aligned} \tag{6.58}$$

$$\text{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-k+2,i}) =$$

$$= - \sum_w n^{-1}_{e-w-k+2} \pi^2_{e-w-k+2,k} p_{e-w-k+2,k} p_{e-w-k+2,i} \tag{6.59}$$

$$\text{var}(\varepsilon_{e-L-k+2,i}) =$$

$$= \sum_w n^{-1}_{e-w-k+2} \pi^2_{e-w-k+2,k} p_{e-w-k+2,i} q_{e-w-k+2,i} \tag{6.60}$$

$$\text{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-k+2,j}) =$$

$$= - \sum_w n^{-1}_{e-w-k+2} \pi^2_{e-w-k+2,k} p_{e-w-k+2,i} p_{e-w-k+2,j} \tag{6.61}$$

The substitution of λ's, (6.58), (6.59), and (6.60) and (6.61) into (6.57) gives the variance of r_k as

$$\text{var}(r_k) = (1 - \sum_i \sum_u \pi_{e-u-k+2,k} p_{e-u-k+2,i})^{-4}$$

$$\times [(1 - \sum_i \sum_u \pi_{e-u-k+2,k} p_{e-u-k+2,i})^2$$

$$\times (\sum_w n^{-1}_{e-w-k+2} \pi^2_{e-w-k+2,k} p_{e-w-k+2,k} q_{e-w-k+2,k})$$

$$+ 2(\sum_u \pi_{e-u-k+2,k} p_{e-u-k+2,k})(1 - \sum_i \sum_v \pi_{e-v-k+2,k} p_{e-v-k+2,i})$$

$$\times \left(- \sum_i \sum_w n^{-1}_{e-w-k+2}\, \pi^2_{e-w-k+2,k}\, p_{e-w-k+2,k}\, p_{e-w-k+2,i}\right)$$

$$+ \left(\sum_u \pi_{e-u-k+2,k}\, p_{e-u-k+2,k}\, p_{e-u-k+2,k}\right)^2$$

$$\times \left(\sum_i \sum_w n^{-1}_{e-w-k+2}\, \pi^2_{e-w-k+2,k}\, p_{e-w-k+2,i}\, q_{e-w-k+2,i}\right.$$

$$\left.- \sum_{i \neq j} \sum_w n^{-1}_{e-w-k+2}\, \pi^2_{e-w-k+2,k}\, p_{e-w-k+2,i}\, p_{e-w-k+2,j}\right)\Big] \qquad (6.62)$$

The multiplication of the terms in the numerator of (6.62) results in:

$$\mathrm{var}(r_k) = \left(1 - \sum_i \sum_u \pi_{e-u-k+2,k}\, p_{e-u-k+2,i}\right)^{-4}$$

$$\times \Big[\sum_w n^{-1}_{e-w-k+2}\, \pi^2_{e-w-k+2,k}\, p_{e-w-k+2,k}\, q_{e-w-k+2,k}$$

$$- 2\sum_{iuw}\sum\sum n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi^2_{e-w-k+2,k}\, p_{e-u-k+2,i}$$

$$\times\, p_{e-w-k+2,k}\, q_{e-w-k+2,k}$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^2_{e-w-k+2,k}$$

$$\times\, p_{e-u-k+2,i}\, p_{e-v-k+2,j}\, p_{e-w-k+2,k}\, q_{e-w-k+2,k}$$

$$- 2\sum_{iuw}\sum\sum n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi^2_{e-w-k+2,k}\, p_{e-u-k+2,k}$$

$$\times\ p_{e-w-k+2,k}\ p_{e-w-k+2,i}$$

$$+\ 2\sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-k+2}\ \pi_{e-u-k+2,k}\ \pi_{e-v-k+2,k}\ \pi^2_{e-w-k+2,k}$$

$$\times\ p_{e-u-k+2,k}\ p_{e-v-k+2,j}\ p_{e-w-k+2,i}\ p_{e-w-k+2,k}$$

$$+\ \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-k+2}\ \pi_{e-u-k+2,k}\ \pi_{e-v-k+2,k}\ \pi^2_{e-w-k+2,k}$$

$$\times\ p_{e-v-k+2,k}\ p_{e-w-k+2,i}\ q_{e-w-k+2,i}$$

$$-\ \sum_{i\neq j}\sum_{uvw}\sum\sum\sum n^{-1}_{e-w-k+2}\ \pi_{e-u-k+2,k}\ \pi_{e-v-k+2,k}\ \pi^2_{e-w-k+2,k}$$

$$\times\ p_{e-u-k+2,k}\ p_{e-v-k+2,k}\ p_{e-w-k+2,i}\ p_{e-w-k+2,j}\,] \qquad (6.63)$$

After further simplification (6.63) can be rewritten as

$$\mathrm{var}(r_k) = (1 - \sum_i \sum_u \pi_{e-u-k+2,k}\ p_{e-u-k+2,i})^{-4}$$

$$\times\ [\sum_w n^{-1}_{e-w-k+2}\ \pi^2_{e-w-k+2,k}\ p_{e-w-k+2,k}\ q_{e-w-k+2,k}$$

$$-\ 2\sum_{iuw}\sum\sum n^{-1}_{e-w-k+2}\ \pi_{e-u-k+2,k}\ \pi^2_{e-w-k+2,k}\ p_{e-u-k+2,i}\ p_{e-w-k+2,k}$$

$$+\ \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-k+2}\ \pi_{e-u-k+2,k}\ \pi_{e-v-k+2,k}\ \pi^2_{e-w-k+2,k}$$

$$\times p_{e-u-k+2,k}\, p_{e-v-k+2,k}\, p_{e-w-k+2,i}$$

$$+ 2\sum_{i}\sum_{u}\sum_{w} n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi^{2}_{e-w-k+2,k}\, p_{e-w-k+2,k}$$

$$\times(p_{e-w-k+2,k}\, p_{e-u-k+2,i} - p_{e-u-k+2,k}\, p_{e-w-k+2,i})$$

$$+ \sum_{i}\sum_{j}\sum_{u}\sum_{v}\sum_{w} n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^{2}_{e-w-k+2,k}$$

$$\times p_{e-u-k+2,i}\, p_{e-v-k+2,j}\, p_{e-w-k+2,k}$$

$$+ \sum_{i}\sum_{j}\sum_{u}\sum_{v}\sum_{w} n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^{2}_{e-w-k+2,k}$$

$$\times(p_{e-w-k+2,k}\, p_{e-v-k+2,j}(p_{e-u-k+2,k}\, p_{e-w-k+2,i}$$

$$-p_{e-w-k+2,k}\, p_{e-u-k+2,i}) + p_{e-u-k+2,k}\, p_{e-w-k+2,i}$$

$$\times(p_{e-w-k+2,k}\, p_{e-v-k+2,j} - p_{e-v-k+2,k}\, p_{e-w-k+2,j}))] \qquad (6.64)$$

Of special interest here is the case in which all vintage groups are subject to the same mortality characteristic, i.e.,

$$p_{e-w-k+2,j} = p_{e-u-k+2,j} \equiv p_j,$$

for all possible values of u, w, and j.

The variance of r_k is then reduced to:

$$\mathrm{var}(r_k) = (1 - \sum_i \sum_w \pi_{e-u-k+2,k}\, p_i)^{-4}$$

$$[p_k\, q_k \sum_w n^{-1}_{e-w-k+2}\, \pi^2_{e-w-k+2,k} - 2\sum_{iuw}\sum\sum n^{-1}_{e-w-k+2}$$

$$\times\, \pi_{e-u-k+2,k}\, \pi^2_{e-w-k+2,k}\, p_k\, p_i$$

$$+ \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^2_{e-w-k+2,k}\, p_k^2\, p_i$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^2_{e-w-k+2,k}\, p_k\, p_i\, p_j]$$

But

$$\sum_{u=1}^{L} \pi_{e-u-k+2,k} = \sum_{v=1}^{L} \pi_{e-v-k+2,k} = 1$$

Hence,

$$\mathrm{var}(r_k) = (1 - \sum_i p_i)^{-4} (\sum_w n^{-1}_{e-w-k+2}\, \pi^2_{e-w-k+2,k})$$

$$\times (p_k\, q_k - 2p_k \sum_i p_i + p_k^2 \sum_i p_i + p_k(\sum_i p_i)^2) \qquad (6.65)$$

The variance of r_k may be written as

$$\mathrm{var}(r_k) = (\sum_w n^{-1}_{e-w-k+2}\, \pi^2_{e-w-k+2,k}) \frac{(R_k - R_k^2)}{(1 - \sum_i p_i)} \qquad (6.66)$$

where

R_k is defined as in (5.26)

Suppose that all vintage groups have the same size, say $n_{e-w-k+2} =$ n for possible value w, then (6.66) is given by

$$\mathrm{var}(r_k) = \frac{\dfrac{(R_k - R_k)^2}{k-1}}{n\ L(1 - \sum_{i=1} p_i)} \tag{6.67}$$

D. L-year Experience Band Based on Item Value

So far, the models were derived based upon item counts. Since most of industrial property accounts are kept in terms of units of dollars rather than item counts hence, it is appropriate to consider the model in terms of dollars (item values).

This section concentrates on modeling based on item value. It can be shown that essentially the basic formulas for the retirement ratios and the corresponding variance and covariance remain the same as for the model based on item counts. The variance and covariance are slightly changed and the meanings of π's and p's are not changed.

1. Derivation of observed retirement ratios

From (6.35), the retirement ratios for the k^{th} age interval can be expressed as

$$r_k = \frac{\sum_w a_{e-w-k+2}\ n_{e-w-k+2,k}}{\sum_w a_{e-w-k+2}\ n_{e-w-k+2} - \sum_i \sum_w a_{e-w-k+2}\ n_{e-w-k+2,i}} \tag{6.68}$$

If both numerator and denumerator of (6.68) are divided by $\sum_{w=1}^{L} n_{e-w-k+2}$, then:

$$r_k = \frac{\sum_w a_{e-w-k+2} \frac{n_{e-w-k+2}}{(\sum_w n_{e-w-k+2})} \cdot \frac{n_{e-2-k+2,k}}{n_{e-w-k+2}}}{\sum_w a_{e-w-k+2} \frac{n_{e-w-k+2}}{(\sum_w n_{e-w-k+2})} - \sum_{iw}\sum a_{e-w-k+2} \frac{n_{e-w-k+2}}{(\sum_w n_{e-w-k+2})} \cdot \frac{n_{e-w-k+2,i}}{n_{e-w-k+2}}}$$

From the definitions of π's and $\hat{p}$'s, r_k may be written as

$$r_k = \frac{\sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k}\, \hat{p}_{e-w-k+2,k}}{\sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k} - \sum_{iw}\sum a_{e-w-k+2}\, \pi_{e-w-k+2,k}\, \hat{p}_{e-w-k+2,i}} \tag{6.69}$$

In terms of λ's and ε's, (6.69) can be expressed as

$$r_k = \frac{\lambda_{e-L-k+2,k} + \varepsilon_{e-L-k+2,k}}{\lambda_{*k} - \sum_i \lambda_{e-L-k+2,i} - \sum_i \varepsilon_{e-L-k+2,i}} \tag{6.70}$$

where

$$\lambda_{*k} = \sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k}$$

$$\varepsilon_{e-L-k+2,i} = \sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k}(\hat{p}_{e-w-k+2,i} - p_{e-w-k+2,i})$$

$$\lambda_{e-L-k+2,i'} = \sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k}\, p_{e-w-k+2,i}$$

Equation (6.70) can be approximated by linear order terms of the

Taylor series:

$$r_k = \phi_{e-L-k+2,k} + \frac{\phi_{e-L-k+2,k}}{\lambda_{e-L-k+2,k}} \varepsilon_{e-L-k+2,k}$$

$$+ \frac{\phi_{e-L-k+2,k}}{(\lambda_{*k} - \lambda^{\circ}_{e-L-k+2,k})} \varepsilon^{\circ}_{e-L-k+2,k} \tag{6.71}$$

where

$$\phi_{e-L-k+2,k} = \frac{\lambda_{e-L-k+2,k}}{(\lambda_{*k} - \lambda^{\circ}_{e-L-k+2,k})}$$

$$\lambda^{\circ}_{e-L-k+2,k} = \sum_{i=1}^{k-1} \lambda_{e-L-k+2,i}$$

and

$$\varepsilon^{\circ}_{e-L-k+2,k} = \sum_{i=1}^{k-1} \varepsilon_{e-L-k+2,i} \; .$$

2. Derivation of large-sample covariance of r_k and r_ℓ

As in the case of formula (6.49), the covariance of retirement ratios that are based on item values can be calculated by the following formula.

$$\text{cov}(r_k, r_\ell) = (\lambda_{*k} - \sum_i \lambda_{e-L-k+2,i})^{-2} (\lambda_{*\ell} - \sum_j \lambda_{e-L-\ell+2,j})^{-2}$$

$$\times [(\lambda_{*k} - \sum_i \lambda_{e-L-k=2,i})(\lambda_{*\ell} - \sum_j \lambda_{e-L-k+2,j})$$

$$\times \text{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,\ell})$$

$$+ \lambda_{e-L-\ell+2,\ell}(\lambda_{*k} - \sum_i \lambda_{e-L-k+2,i})(\mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,k})$$

$$+ \sum_{j\neq k} \mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,j}))$$

$$+ (\lambda_{e-L-k+2,k})(\lambda_{*\ell} - \sum_j \lambda_{e-L-\ell+2,j}) \sum_i \mathrm{cov}(\varepsilon_{e-L-\ell+2,\ell}, \varepsilon_{e-L-k+2,i})$$

$$+ \lambda_{e-L-k+2,k} \lambda_{e-L-\ell+2,\ell}(\sum_i \mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,i})$$

$$+ \sum_{i\neq j}\sum \mathrm{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,j}))] \qquad (6.72)$$

for $i = 1, 2, \ldots, k-1$; $j = 1, 2, \ldots, \ell - 1$.

Again, if it is assumed that each vintage group included in the study dies according to multinomial distribution with parameter $n_{e-w-k+2}$ and $p_{e-w-k+2,i}$, for $i = 1, 2, \ldots, N$ then covariances and variances of ε's in (6.72) can be estimated by formulas (6.43) and (6.44):

$$\mathrm{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,\ell})$$

$$= - \sum_w n^{-1}_{e-w-\ell+2} a^2_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-2-\ell+2,\ell}$$

$$\times p_{e-w-\ell+2,k} p_{e-w-\ell+2,\ell} \qquad (6.73)$$

$$\text{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,k}) =$$

$$= \sum_{w} n^{-1}_{e-w-\ell+2} a^{2}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell}$$

$$\times p_{e-w-\ell+2,k} q_{e-w-\ell+2,k} \tag{6.74}$$

$$\sum_{j\neq k} \text{cov}(\varepsilon_{e-L-k+2,k}, \varepsilon_{e-L-\ell+2,j}) =$$

$$= - \sum_{j\neq k} \sum_{w} n^{-1}_{e-w-\ell+2} a^{2}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k}$$

$$\times p_{e-w-\ell+2,j} \tag{6.75}$$

$$\sum_{i} \text{cov}(\varepsilon_{e-L-\ell+2,\ell}, \varepsilon_{e-L-k+2,i}) =$$

$$= - \sum_{i} \sum_{w} n^{-1}_{e-w-\ell+2} a^{2}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,\ell}$$

$$\times p_{e-w-\ell+2,i} \tag{6.76}$$

$$\sum_{i} \text{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,i}) =$$

$$= + \sum_{i} \sum_{w} n^{-1}_{e-w-\ell+2} a^{2}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,i}$$

$$\times q_{e-w-\ell+2,i} \tag{6.77}$$

$$\sum_{i \neq j}\sum \operatorname{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-\ell+2,j}).$$

$$= - \sum_{i \neq j}\sum \sum_{w} n^{-1}_{e-w-\ell+2} a^{2}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,i}$$

$$\times p_{e-w-\ell+2,j} \qquad (6.78)$$

With substitution of λ's, (6.73), (6.74), (6.75), (6.76), (6.77) and (6.78) into (6.72) yields

$$\operatorname{cov}(r_k, r_\ell) =$$

$$= (\sum_{s} a_{e-w-k+2} \pi_{e-s-k+2,k} - \sum_{iu}\sum a_{e-u-k+2} \pi_{e-u-k+2,k} p_{e-u-k+2,i})^{-2}$$

$$\times (\sum_{t} a_{e-t-\ell+2} \pi_{e-t-\ell+2,\ell} - \sum_{jv}\sum a_{e-v-\ell+2} \pi_{e-v-\ell+2,\ell} p_{e-v-\ell+2,j})^{-2}$$

$$\times [(\sum_{s} a_{e-s-k+2} \pi_{e-s-k+2,k} - \sum_{iu}\sum a_{e-u-k+2} \pi_{e-u-k+2,k} p_{e-u-k+2,i})$$

$$\times (\sum_{t} a_{e-t-\ell+2} \pi_{e-t-\ell+2,\ell} - \sum_{jv}\sum a_{e-v-\ell+2} \pi_{e-v-\ell+2,\ell} p_{e-v-\ell+2,j})$$

$$\times (- \sum_{w} n^{-1}_{e-w-\ell+2} a^{2}_{e-w-\ell+2} \pi_{e-w-\ell+2,k} \pi_{e-w-\ell+2,\ell} p_{e-w-\ell+2,k} p_{e-w-\ell+2,\ell})$$

$$+ (\sum_{t} a_{e-t-\ell+2} \pi_{e-t-\ell+2,\ell} p_{e-t-\ell+2,\ell})$$

$$\times (\sum_{s} a_{e-s-k+2} \pi_{e-s-k+2,k} - \sum_{iu}\sum a_{e-u-k+2} \pi_{e-u-k+2,k} p_{e-u-k+2,i})$$

$$\times (+\sum_{w} n^{-1}_{e-w-\ell+2}\, a^{2}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}$$

$$\times q_{e-w-\ell+2,k}$$

$$- \sum_{j\neq k} \sum_{w} n^{-1}_{e-w-\ell+2}\, a^{2}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}$$

$$\times p_{e-w-\ell+2,j})$$

$$+ (\sum_{s} a_{e-s-k+2}\, \pi_{e-s-k+2,k}\, p_{e-s-k+2,k})$$

$$\times (\sum_{t} a_{e-t-\ell+2}\, \pi_{e-t-\ell+2,\ell} - \sum\sum_{jv} a_{e-v-\ell+2}\, \pi_{e-v-\ell+2,\ell}\, p_{e-v-\ell+2,j})$$

$$\times (- \sum\sum_{iw} n^{-1}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i})$$

$$+ (\sum_{s} a_{e-s-k+2}\, \pi_{e-s-k+2,k}\, p_{e-s-k+2,k})(\sum_{t} a_{e-t-\ell+2}\, \pi_{e-t-\ell+2,\ell}$$

$$\times p_{e-t-\ell+2,\ell})$$

$$\times (\sum\sum_{iw} n^{-1}_{e-w-\ell+2}\, a^{2}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}$$

$$\times q_{e-w-\ell+2,i}$$

$$- \sum_{i\neq j} \sum_{w} \sum n^{-1}_{e-w-\ell+2}\, a^{2}_{e-w-\ell+2}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,i}\, p_{e-w-\ell+2,j})] \qquad (6.79)$$

When the terms in the numerators are multiplied out, (6.79) can be written as

$$\mathrm{cov}(r_k, r_\ell) =$$

$$= (\sum_s a_{e-s-k+2}\, \pi_{e-s-k+2,k} - \sum_{iu}\sum a_{e-u-k+2}\, \pi_{e-u-k+2,k}\, p_{e-u-k+2,i})^{-2}$$

$$\times (\sum_t a_{e-t-\ell+2}\, \pi_{e-t-k+2,\ell} - \sum_{jv}\sum a_{e-v-\ell+2}\, \pi_{e-v-\ell+2,\ell}\, p_{e-v-\ell+2,j})^{-2}$$

$$[- \sum_{stw}\sum\sum n^{-1}_{e-w-\ell+2}\, a_{e-s-k+2}\, a_{e-t-\ell+2}\, a^2_{e-w-\ell+2}\, \pi_{e-s-k+2,k}$$

$$\times \pi_{e-t-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}\, p_{e-w-\ell+2,\ell}$$

$$+ \sum_{ituw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\, a_{e-t-\ell+2}\, a_{e-u-k+2}\, a^2_{e-w-\ell+2}\, \pi_{e-t-\ell+2,\ell}\, \pi_{e-u-k+2,k}$$

$$\times \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}\, p_{e-w-\ell+2,\ell}\, p_{e-u-k+2,i}$$

$$+ \sum_{jsvw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\, a_{e-s-k+2}\, a_{e-v-\ell+2}\, a^2_{e-w-\ell+2}\, \pi_{e-s-k+2,k}$$

$$\times \pi_{e-v-\ell+2,\ell}\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2,k}$$

$$\times\ p_{e-w-\ell+2,\ell}\, p_{e-v-\ell+2,j}$$

$$- \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\, a_{e-u-k+2}\, a_{e-v-\ell+2}\, a^2_{e-w-\ell+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-\ell+2,\ell}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-u-k+2,i}\ p_{e-v-\ell+2,j}\ p_{e-w-\ell+2,k}\ p_{e-w-\ell+2,\ell}$$

$$+\ \sum_{stw}\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-t-\ell+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-t-\ell+2,\ell}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-t-\ell+2,\ell}\ p_{e-2-\ell+2,k}\ q_{e-w-\ell+2,k}$$

$$-\ \sum_{ituw}\sum\sum\sum n^{-1}_{e-w-\ell+w}\ a_{e-t-\ell+2}\ a_{e-u-k+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-t-\ell+2,\ell}\ \pi_{e-u-k+2,k}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-t-\ell+2,\ell}\ p_{e-u-k+2,i}\ p_{e-w-\ell+2,k}\ q_{e-w-\ell+2,k}$$

$$-\ \sum_{\substack{j\\ j\neq k}}\ \sum_{s}\ \sum_{tw}\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-t-\ell+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-t-\ell+2,\ell}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-t-\ell+2,\ell}\ p_{e-w-\ell+2,k}\ p_{e-w-\ell+2,j}$$

$$+\ \sum_{ijtuw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-t-\ell+2}\ a_{e-u-k+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-t-\ell+2,\ell}\ \pi_{e-u-k+2,k}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-t-\ell+2,\ell}\ p_{e-u-k+2,i}\ p_{e-w-\ell+2,k}\ p_{e-w-\ell+2,j}$$

$$-\ \sum_{istw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-t-\ell+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-t-\ell+2,\ell}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-s-k+2,k}\ p_{e-w-\ell+2,\ell}\ p_{e-w-\ell+2,i}$$

$$+\ \sum_{ijsvw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-v-\ell+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-v-\ell+2,\ell}$$

$$\times \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-s-k+2,k}\ p_{e-w-\ell+2,\ell}\ p_{e-w-\ell+2,i}$$

$$\times \pi_{e-v-\ell+2,j}$$

$$+ \sum_{istw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-t-\ell+2}\ a^2_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-t-\ell+2,\ell}$$

$$\times \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-s-k+2,k}\ p_{e-t-\ell+2,\ell}\ p_{e-w-\ell+2,i}$$

$$\times q_{e-w-\ell+2,i}$$

$$- \sum_{\substack{i\,jstw \\ i=j}}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-t-\ell+2}\ a^2_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-t-\ell+2,\ell}$$

$$\times \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-s-k+2,k}\ p_{e-t-\ell+2,\ell}\ p_{e-w-\ell+2,i}$$

$$\times p_{e-w-\ell+2,j}] \qquad (6.80)$$

After considerable simplification, (6.80) may be presented as:

$$\mathrm{cov}(r_k,\ r_\ell) =$$

$$= (\sum_s a_{e-s-k+2}\ \pi_{e-s-k+2,k} - \sum_{iu}\sum a_{e-u-k+2}\ \pi_{e-u-k+2,k}\ p_{e-u-k+2,i})^{-2}$$

$$\times (\sum_t a_{e-t-\ell+2}\ \pi_{e-t-\ell+2,\ell} - \sum_{jv}\sum a_{e-v-\ell+2}\ \pi_{e-v-\ell+2,\ell}\ p_{e-v-\ell+2,j})^{-2}$$

$$\times \ [\sum_{stw}\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-t-\ell+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-t-\ell+2,\ell}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-w-\ell+2,k}(p_{e-t-\ell+2,\ell} - p_{e-w-\ell+2,\ell})$$

$$+ \sum_{ituw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-t-\ell+2}\ a_{e-u-k+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-t-\ell+2,\ell}$$

$$\times\ \pi_{e-u-k+2,k}\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-w-\ell+2,k}\ p_{e-u-k+2,i}$$

$$\times\ (p_{e-w-\ell+2,\ell} - p_{e-t-\ell+2,\ell})$$

$$+ \sum_{jsvw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-s-k+2}\ a_{e-v-\ell+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-s-k+2,k}\ \pi_{e-v-\ell+2,\ell}$$

$$\times\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-w-\ell+2,k}(p_{e-w-\ell+2,\ell}\ p_{e-v-\ell+2,j} -$$

$$- p_{e-v-\ell+2,\ell}\ p_{e-w-\ell+2,j})$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-\ell+2}\ a_{e-u-k+2}\ a_{e-v-\ell+2}\ a^{2}_{e-w-\ell+2}\ \pi_{e-u-k+2,k}$$

$$\times\ \pi_{e-v-\ell+2,\ell}\ \pi_{e-w-\ell+2,k}\ \pi_{e-w-\ell+2,\ell}\ p_{e-w-\ell+2,k}\ p_{e-u-k+2,i}$$

$$\times\ (p_{e-v-\ell+2,\ell}\ p_{e-w-\ell+2,j} - p_{e-w-\ell+2,\ell}\ p_{e-v-\ell+2,j})$$

$$+ \sum_{ijsvw}\Sigma\Sigma\Sigma\Sigma\; n^{-1}_{e-w-\ell+2}\; a_{e-s-k+2}\; a_{e-v-\ell+2}\; a^{2}_{e-w-\ell+2}\; \pi_{e-s-k+2,k}$$

$$\times\; \pi_{e-v-\ell+2,\ell}\; \pi_{e-w-\ell+2,k}\; \pi_{e-w-\ell+2,\ell}\; p_{e-s-k+2,k}\; p_{e-w-\ell+2,i}$$

$$\times\; (p_{e-w-\ell+2,\ell}\; p_{e-v-\ell+2,j} - p_{e-v-\ell+2,\ell}\; p_{e-w-\ell+2,j})] \qquad (6.81)$$

for $i = 1, 2, \ldots, k-1$; $j = 1, 2, \ldots, \ell-1$.

$s, t, u, v, = 1, 2, \ldots, L$; $w = 1, 2, \ldots, L + k - \ell$.

Under the condition that all vintage groups have the same life distribution,

$$p_{e-v-\ell+2,j} = p_{e-w-\ell+2,j} \equiv p_j \text{ for all v, w, and j.}$$

Then, it can be easily observed that $\text{cov}(r_k, r_\ell) = 0$.

3. Derivation of large-sample variance of r_k

In a manner similar to that used for equation (6.57), the estimates of variance of retirement ratios for each age interval can be computed by the following formula.

$$\text{var}(r_k) = (\lambda_{*k} - \sum_i \lambda_{e-L-k+2,i})^{-4}$$

$$\times\; [(\lambda_{*k} - \sum_i \lambda_{e-L-k+2,i})^2\; \text{var}(\varepsilon_{e-L-k+2,k})$$

$$\times\; 2\lambda_{e-L-k+2,k}(\lambda_{*k} - \sum_w \lambda_{e-L-k+2,i})\; \sum_i \text{cov}(\varepsilon_{e-L-k+2,k},\; \varepsilon_{e-L-k+2,i})$$

$$+ \lambda^2_{e-L-k+2,k}(\sum_i \text{var}(\varepsilon_{e-L-k+2,i}) + \sum_{i \neq j}\sum \text{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-k+2,j}))] \quad (6.82)$$

According to Lemma 1 (Chiang, 1960a), the number of units retired from each vintage group has the multinomial distribution with parameters $n_{e-w-k+2}$ and $p_{e-w-k+2,i}$. Therefore, the variances and covariances of ε's in (6.82) can be derived from (6.44) and (6.43):

$$\text{var}(\varepsilon_{e-L-k+2,i}) =$$

$$= \sum_w n^{-1}_{e-w-k+2}\, a^2_{e-w-k+2}\, \pi^2_{e-w-k+2,k}\, p_{e-w-k+2,i}\, q_{e-w-k+2,i}$$

and

$$\text{cov}(\varepsilon_{e-L-k+2,i}, \varepsilon_{e-L-k+2,j}) =$$

$$= - \sum_w n^{-1}_{e-w-k+2}\, a^2_{e-w-k+2}\, \pi^2_{e-w-k+2,k}\, p_{e-w-k+2,i}\, p_{e-w-k+2,j}$$

for $w = 1, 2, \ldots L$

$i, j = 1, 2, \ldots, N$

Upon the substitution of λ's, the variances and covariances of ε's into (6.82), variance of r_k may be written as

$$\text{var}(r_k) = (\sum_s a_{e-s-k+2}\, \pi_{e-s-k+2,k} - \sum_{iv}\sum a_{e-v-k+2}\, \pi_{e-v-k+2,k}\, p_{e-v-k+2,i})^{-4}$$

$$\times [(\sum_s a_{e-s-k+2}\, \pi_{e-s-k+2,k} - \sum_{iv}\sum a_{e-v-k+2}\, \pi_{e-v-k+2,k}\, p_{e-v-k+2,i})^2$$

$$\times \left(\sum_{w} n^{-1}_{e-w-k+2}\, a^{2}_{e-w-k+2}\, \pi^{2}_{e-w-k+2,k}\, p_{e-w-k+2,k}\, q_{e-w-k+2,k}\right)$$

$$\times\ 2\left(\sum_{u} a_{e-u-k+2}\, \pi_{e-u-k+2,k}\, p_{e-u-k+2,k}\right)\left(\sum_{s} a_{e-s-k+2}\, \pi_{e-s-k+2,k}\right.$$

$$\left. - \sum_{iv}\sum a_{e-v-k+2}\, \pi_{e-v-k+2,k}\, p_{e-v-k+2,i}\right)$$

$$\times \left(- \sum_{iw}\sum n^{-1}_{e-w-k+2}\, a^{2}_{e-w-k+2}\, \pi^{2}_{e-w-k+2,k}\, p_{e-w-k+2,k}\, p_{e-2-k+2,i}\right)$$

$$+ \left(\sum_{u} a_{e-u-k+2}\, \pi_{e-u-k+2,k}\, p_{e-u-k+2,k}\right)^{2}$$

$$\times \left(\sum_{iw}\sum n^{-1}_{e-w-k+2}\, a^{2}_{e-w-k+2}\, \pi^{2}_{e-w-k+2,k}\, p_{e-w-k+2,i}\, q_{e-w-k+2,i}\right.$$

$$\left. - \sum_{i \neq j}\sum \sum_{w} n^{-1}_{e-w-k+2}\, a^{2}_{e-w-k+2}\, \pi^{2}_{e-w-k+2,k}\, p_{e-w-k+2,i}\, p_{e-w-k+2,j}\right)\Big] \tag{6.83}$$

for $i, j = 1, 2, \ldots, k - 1$;

$s, u, v, 2 = 1, 2, \ldots, L.$

When the terms in the numerator of (6.83) are multiplied out it yields

$$\mathrm{var}(r_k) = \left(\sum_{s} a_{e-s-k+2}\, \pi_{e-s-k+2,k} - \sum_{iv}\sum a_{e-v-k+2}\, \pi_{e-v-k+2,k}\, p_{e-v-k+2,i}\right)^{-4}$$

$$\times \Big[\sum_{stw}\sum\sum n^{-1}_{e-w-\ell+2}\, a_{e-s-k+2}\, a_{e-t-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-s-k+2,k}\, \pi_{e-t-k+2,k}$$

$$\times\ \pi^{2}_{e-w-k+2,k}\, p_{e-w-k+2,k}\, q_{e-w-k+2,k}$$

$$- 2\sum_{isvw}\sum\sum\sum n^{-1}_{e-w-\ell+2}\, a_{e-s-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-s-k+2,k}$$

$$\times \pi_{e-v-k+2,k}\, \pi^{2}_{e-w-k+2,k}\, p_{e-w-k+2,k}\, q_{e-w-k+2,k}\, p_{e-v-k+2,i}$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}$$

$$\times \pi^{2}_{e-w-k+2,k}\, p_{e-u-k+2,i}\, p_{e-v-k+2,j}\, p_{e-w-k+2,k}\, q_{e-w-k+2,k}$$

$$- 2\sum_{isuw}\sum\sum\sum n^{-1}_{e-w-k+2}\, a_{e-s-k+2}\, a_{e-u-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-s-k+2,k}\, \pi_{e-u-k+2,k}$$

$$\times \pi^{2}_{e-w-k+2,k}\, p_{e-u-k+2,k}\, p_{e-w-k+2,k}\, p_{e-w-k+2,i}$$

$$+ 2\sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}$$

$$\times \pi^{2}_{e-w-k+2,k}\, p_{e-u-k+2,k}\, p_{e-w-k+2,k}\, p_{e-w-k+2,i}\, p_{e-v-k+2,j}$$

$$+ \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}$$

$$\times \pi^{2}_{e-w-k+2,k}\, p_{e-u-k+2,k}\, p_{e-v-k+2,k}\, p_{e-w-k+2,i}\, q_{e-w-k+2,i}$$

$$- \sum_{i\neq j}\sum_{u}\sum_{v}\sum_{w} n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}$$

$$\times \pi^{2}_{e-w-k+2,k}\, p_{e-u-k+2,k}\, p_{e-v-k+2,k}\, p_{e-w-k+2,i}\, p_{e-w-k+2,j}] \qquad (6.84)$$

After considerable simplification, (6.84) may be presented as

$$\mathrm{var}(r_k) = \Big(\sum_s a_{e-s-k+2}\,\pi_{e-s-k+2,k} - \sum_{iv}\sum a_{e-v-k+2}\,\pi_{e-v-k+2,k}\,p_{e-v-k+2,i}\Big)^{-4}$$

$$\times \Big[\sum_{stw}\sum\sum n^{-1}_{e-w-k+2}\,a_{e-s-k+2}\,a_{e-t-k+2}\,a^2_{e-w-k+2}\,\pi_{e-s-k+2,k}\,\pi_{e-t-k+2,k}$$

$$\times \pi^2_{e-w-k+2,k}\,p_{e-w-k+2,k}\,q_{e-w-k+2,k}$$

$$+ \sum_{iuvw}\sum\sum\sum n^{-1}_{e-w-k+2}\,a_{e-u-k+2}\,a_{e-v-k+2}\,a^2_{e-w-k+2}\,\pi_{e-u-k+2,k}\,\pi_{e-v-k+2,k}$$

$$\times \pi^2_{e-w-k+2,k}\,p_{e-u-k+2,k}\,p_{e-v-k+2,k}\,p_{e-w-k+2,i}$$

$$- 2\sum_{isvw}\sum\sum\sum n^{-1}_{e-w-k+2}\,a_{e-s-k+2}\,a_{e-v-k+2}\,a^2_{e-w-k+2}\,\pi_{e-s-k+2,k}$$

$$\times \pi_{e-v-k+2,k}\,\pi^2_{e-w-k+2,k}\,p_{e-w-k+2,k}\,p_{e-v-k+2,i}$$

$$+ 2\sum_{isvw}\sum\sum\sum n^{-1}_{e-w-k+2}\,a_{e-s-k+2}\,a_{e-v-k+2}\,a^2_{e-w-k+2}\,\pi_{e-s-k+2,k}\,\pi_{e-v-k+2,k}$$

$$\times \pi^2_{e-w-k+2,k}\,p_{e-w-k+2,k}(p_{e-w-k+2,k}\,p_{e-v-k+2,i} - p_{e-u-k+2,k}$$

$$\times p_{e-w-k+2,i})$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-k+2}\,a_{e-u-k+2}\,a_{e-v-k+2}\,a^2_{e-w-k+2}\,\pi_{e-u-k+2,k}\,\pi_{e-v-k+2,k}$$

$$\times \pi^2_{e-w-k+2,k}\,p_{e-w-k+2,k}\,p_{e-u-k+2,i}\,p_{e-v-k+2,j}$$

$$+ \sum_{ijuvw}\sum\sum\sum\sum n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}$$

$$\times\, \pi^{2}_{e-w-k+2,k}\{p_{e-w-k+2,k}\, p_{e-v-k+2,j}(p_{e-u-k+2,k}\, p_{e-w-k+2,i}$$

$$-\, p_{e-u-k+2,i}\, p_{e-w-k+2,k}) + p_{e-u-k+2,k}\, p_{e-w-k+2,i}$$

$$\times\, (p_{e-w-k+2,k}\, p_{e-v-k+2,j} - p_{e-v-k+2,k}\, p_{e-w-k+2,j})\}] \qquad (6.85)$$

When all vintage groups are subject to the same mortality characteristic, i.e.,

$$p_{e-u-k+2,j} = p_{e-v-k+2,j} \equiv p_j$$

for all possible values of u, v and j, then the estimate of the variance of r_k is much simplified:

$$\mathrm{var}(r_k) = (\sum_s a_{e-s-k+2}\, \pi_{e-s-k+2,k}(1 - \sum_i p_i))^{-4}$$

$$\times\, [\sum_{stw}\sum\sum n^{-1}_{e-w-k+2}\, a_{e-s-k+2}\, a_{e-t-k+2}\, a^{2}_{e-w-k+2}$$

$$\times\, \pi_{e-s-k+2,k}\, \pi_{e-t-k+2,k}\, \pi^{2}_{e-w-k+2,k}\, p_k\, q_k$$

$$+ \sum_{uvw}\sum\sum n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}$$

$$\times\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^{2}_{e-w-k+2,k}\, p_k^{2}(\sum_i p_i)$$

$$- 2 \sum_{suw}\sum\sum n^{-1}_{e-w-k+2}\, a_{e-s-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}$$

$$\times\, \pi_{e-w-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^{2}_{e-w-k+2,k}\, p_k(\sum_i p_i)$$

$$+ \sum_{uvw}\sum\sum n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}$$

$$\times\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^{2}_{e-w-k+2,k}\, p_k(\sum_i p_i)^2] \qquad (6.86)$$

Equation (6.86) may be written as

$$\mathrm{var}(r_k) = (\sum_s a_{e-s-k+2}\, \pi_{e-s-k+2,k}(1 - \sum_i p_i))^{-4}$$

$$\times\, [\sum_{uvw}\sum\sum n^{-1}_{e-w-k+2}\, a_{e-u-k+2}\, a_{e-v-k+2}\, a^{2}_{e-w-k+2}$$

$$\times\, \pi_{e-u-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^{2}_{e-w-k+2,k}$$

$$\times\, (p_k\, q_k - 2p_k \sum_i p_i + p_k{}^2 \sum_i p_i + p_k(\sum p_i)^2)]$$

or

$$\mathrm{var}(r_k) = (\sum_s a_{e-s-k+2}\, \pi_{e-s-k+2,k})^{-2}$$

$$\times\, (\sum_w n^{-1}_{e-w-k+2}\, a^{2}_{e-w-k+2}\, \pi^{2}_{e-w-k+2,k}) \cdot \frac{(R_k - R_k{}^2)}{(1 - \sum_i p_i)} \qquad (6.87)$$

for $i = 1, 2, \ldots, k-1$

$s, w, = 1, 2, \ldots, L.$

Of special interest here is the case in which costs per unit from all vintage groups and all vintage sizes are the same, say a and n, respectively. Then, it can be shown that (6.87) is simplified to:

$$\mathrm{var}(r_k) = \frac{(R_k - R_k^2)}{\mathrm{Ln}(1 - \sum_{i=1}^{k-1} p_i)} \tag{6.88}$$

where

R_k - the true hazard rate

$$= \frac{p_k}{(1 - \sum_{i=1}^{k-1} p_i)}$$

VII. THE DISCRETE MODEL UNDER GEOMETRIC CONDITIONAL DISTRIBUTIONS

This chapter deals mainly with geometric distributions. It basically consists of three sections. The first presents derivations of the estimates of variances and covariances of hazard rates for long-lived property when value or vintage groups are assumed to follow geometric distributions. The second demonstrates the relative variance efficiency of the ordinary least square (OLS) estimator to weighted least square (WLS) estimator for the estimates of the true average hazard rates.

In the third section, an evaluation is made of the bias of the estimators derived in part two.

A. Geometric Distribution

Random variable x is defined to have a geometric distribution if the density function of x is given by

$$f(x) = P(1 - P)^{x}$$
$$= 0 \qquad \text{otherwise } x = 0, 1, 2$$

The geometric distribution has interesting features, namely,

1. it is parameterized by single parameter p,
2. the hazard rate associated with it is constant, and
3. it can be used to represent long-lived property when p is small.

The equivalent of the geometric distribution in the continuous case is negative exponential distribution.

In engineering valuation, the constant hazard rates can be interpreted as describing the situation in which the portion of the property being retired in any year is independent of age.

Some industrial properties have long-lived distributions, hence, their retirement experience can be modeled by geometric distribution with small p, where p represents the true probability of units or dollars retired during any age interval.

B. Variance-covariance Structures of Hazard Rates for Long Expected Life

1. Property groups classified by value and life

For the first k periods, the retirement experience from the property group of value a_s is assumed to follow a geometric distribution having parameter p_{a_s}, i.e.,

$$p_{a_{si}} = p_{a_s}(1 - p_{a_s})^i, \text{ for } i = 1, 2, \ldots, k.$$

For small p_{a_s},

$$p_{a_{si}} \doteq p_{a_s}(1 - i\, p_{a_s}), \text{ for } i = 1, 2, \ldots, k. \tag{7.1}$$

It is important to note here that only in the first k periods the retirement experience is assumed to follow the geometric life distributions. The remaining life of a property group may follow some other kinds of life distributions. The advent of technology, management policy, economic conditions, etc. may affect the characteristics of the future life of the property group.

Upon the substitution of condition (7.1) into (5.33) and after further simplification, the approximate covariance of r_k may be written as

$$\text{cov}(r_k, r_\ell) =$$

$$= (\sum_s a_s \pi_{a_s} - \sum_i \sum_r a_r \pi_{a_r} p_{a_r})^{-2} (\sum_u a_u \pi_{a_u} - \sum_j \sum_v a_v \pi_{a_v} p_{a_v})^{-2} \times$$

$$\times [\sum_{sru}\sum\sum \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r})(a_u \pi_{a_u}) p_{a_s}(p_{a_r} - p_{a_s}) + o_3(p_{a_s})]$$

where $o_3(p_{a_s})$ represents the summations of p_{a_s} terms of order of at least three. Considerable simplification may be obtained when the terms of order three are negligible. It can be shown that

$$\text{cov}(r_k, r_\ell) =$$

$$= (\sum_s a_s \pi_{a_s})^{-3} [\sum_s \sum_r \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r}) \times$$

$$\times p_{a_s}(p_{a_r} - p_{a_s})] \tag{7.2}$$

Similarly, the variance of retirement ratios for each age interval under geometric life distributions can be approximated by

$$\mathrm{var}(r_k) = (\sum_s a_s \pi_{a_s})^{-2} \sum_s \frac{(a_s \cdot \pi_{a_s})^2}{n_{a_s}} (p_{a_s} - (k+1) p_{a_s}{}^2)$$

$$+ 2(k-1)(\sum_s a_s \pi_{a_s})^{-3} \sum_s \sum_r \frac{(a_s \pi_{a_s})^2}{n_{a_s}} (a_r \pi_{a_r}) p_{a_s} p_{a_r} \qquad (7.3)$$

2. Property groups classified by vintage and life

For each vintage group which is included in the study, it is assumed for the first k periods, that its retirement experience follows a geometric distribution having parameters $n_{e-w-i+2}$ and $p_{e-w-i+2}$, i.e.,

$$p_{e-w-i+2,i} = p_{e-w-i+2}(1 - p_{e-w-i+2})^i$$

for $w = 1, 2, \ldots, L$

$i = 1, 2, \ldots, k$

For small $p_{e-w-i+2}$,

$$p_{e-w-i+2,i} \doteq p_{e-w-i+2}(1 - i\, p_{e-w-i+2}). \qquad (7.4)$$

Under geometric conditional distributions, approximate covariances of r_k and r_ℓ can be derived as follows.

The substitution of (7.4) for p's, and after further simplification (6.56) may be written as

$$\mathrm{cov}(r_k, r_\ell) = (1 - (k-1) \sum_u \pi_{e-u-k+2,k}\, p_{e-u-k+2})^{-2} \times$$

$$\times (1 - (\ell - 1) \sum_v \pi_{e-v-\ell+2,\ell}\, p_{e-v-\ell+2})^{-2} \times$$

$$\times \ [\sum_u \sum_w n^{-1}_{e-w-\ell+2} \ \pi_{e-w-\ell+2,k} \ \pi_{e-w-\ell+2,\ell} \ \pi_{e-w-\ell+2,\ell} \ \times$$

$$\times \ p_{e-w-\ell+2}(p_{e-u-\ell+2} - p_{e-w-\ell+2}) + 0_3(P)] \tag{7.5}$$

But

$$(1 - (k - 1) \sum_u \pi_{e-u-k+2,k} \ p_{e-u-k+2})^{-2} =$$

$$= 1 + 2(k - 1) \sum_u \pi_{e-u-k+2,k} \ p_{e-u-k+2} + \cdots \tag{7.6}$$

for all k, u.

Upon the substitution of (7.6) into (7.5), and when the terms of order three are negligible covariance of r_k and r_ℓ may be presented as

$$\mathrm{cov}(r_k, \ r_\ell) = \sum_u \sum_w n^{-1}_{e-w-\ell+2} \ \pi_{e-w-\ell+2,k} \ \pi_{e-w-\ell+2,\ell} \ \pi_{e-u-\ell+2,\ell} \ \times$$

$$\times \ p_{e-w-\ell+2}(p_{e-u-\ell+2} - p_{e-w-\ell+2}) \tag{7.7}$$

for u = 1, 2, ... L; w = 1, 2, ..., L + k - ℓ.

Under geometric conditional distributions, estimates variance of r_k can be derived as follows. With the substitution of (7.4) for p's, and after further simplification, (6.64) can be written as

$$\mathrm{var}(r_k) = (1 - (k - 1) \sum_u \pi_{e-u-k+2,k} \ p_{e-u-k+2})^{-4} \ \times$$

$$\times [\sum_w n^{-1}_{e-w-k+2} \pi^2_{e-w-k+2,k} p_{e-w-k+2} q_{e-w-k+2} -$$

$$- 2(k - 1) \sum\sum_{uw} n^{-1}_{e-w-k+2} \pi_{e-u-k+2,k} \pi^2_{e-w-k+2,k} p_{e-w-k+2} p_{e-u-k+2}$$

$$+ 0_3(P)] \qquad (7.8)$$

But

$$(1 - (k - 1) \sum_u \pi_{e-u-2,k} p_{e-u-k+2})^{-4} =$$

$$(1 + 4(k - 1) \sum_u \pi_{e-u-k+2,k} p_{e-u-k+2} + \ldots \qquad (7.9)$$

Upon the substitution of (7.9) into (7.8) and when the terms of order three are negligible, variance of r_k may be presented as

$$\mathrm{var}(r_k) = \sum_w n^{-1}_{e-w-k+2} \pi^2_{e-w-k+2,k} p_{e-w-k+2} q_{e-w-k+2} -$$

$$- 2(k - 1) \sum\sum_{uw} n^{-1}_{e-w-k+2} \pi_{e-u-k+2,k} \pi^2_{e-w-k+2,k} p_{e-u-k+2} p_{e-w-k+2} \qquad (7.10)$$

The above derivations deal with data which are measured on the basis of item counts. Analogously, for the case of data that are kept in units of dollars, under geometric conditional distributions, covariances and variances are estimated:

$$\mathrm{cov}(r_k, r_\ell) = (\sum_u a_{e-u-k+2} \pi_{e-u-k+2,k})^{-2} (\sum_v a_{e-v-\ell+2} \pi_{e-v-\ell+2,\ell})^{-2} \times$$

$$\times \left[\sum_{stw}\sum\sum n^{-1}_{e-w-\ell+2}\, a_{e-s-k+2}\, a_{e-t-k+2}\, a^2_{e-w-\ell+2}\, \pi_{e-s-k+2,k}\, \pi_{e-t-\ell+2,\ell} \times\right.$$

$$\left.\times\, \pi_{e-w-\ell+2,k}\, \pi_{e-w-\ell+2,\ell}\, p_{e-w-\ell+2}(p_{e-t-\ell+2} - p_{e-w-\ell+2})\right], \qquad (7.11)$$

and

$$\mathrm{var}(r_k) = \left(\sum_u a_{e-u-k+2}\, \pi_{e-u-k+2,k}\right)^{-4} \times$$

$$\times \left[\sum_{stw}\sum\sum n^{-1}_{e-w-k+2}\, a_{e-s-k+2}\, a_{e-t-k+2}\, a^2_{e-w-k+2}\, \pi_{e-s-k+2,k}\, \pi_{e-t-k+2,k} \times\right.$$

$$\times\, \pi^2_{e-w-k+2,k}\, p_{e-w-k+2}\, q_{e-w-k+2} -$$

$$- 2(k-1) \sum_{svw}\sum\sum n^{-1}_{e-w-k+2}\, a_{e-s-k+2}\, a_{e-v-k+2}\, a^2_{e-w-k+2} \times$$

$$\left.\times\, \pi_{e-s-k+2,k}\, \pi_{e-v-k+2,k}\, \pi^2_{e-w-k+2,k}\, p_{e-v-k+2}\, p_{e-w-k+2}\right]. \qquad (7.12)$$

C. Variance Efficiency of the Average vs Weighted Average Retirement Ratios

It was mentioned earlier that mortality characteristics for long-lived property may be modeled by geometric distributions. In this case, in view of the hazard rate properties of the geometric distribution, a parameter of special interest is

$$\theta = 1/k \sum_{i=1}^{k} E(r_i) \qquad (7.13)$$

Suppose it is desired to estimate θ. Two kinds of estimators may be used to estimate (7.13), namely,

$$\hat{\theta}_1 = 1/k \sum_{i=1}^{k} r_i \tag{7.14}$$

and

$$\hat{\theta}_2 = \sum_{i=1}^{k} w_i\, r_i \tag{7.15}$$

where the weights, w_i, are chosen to take the variance-covariance structure of the r_i into account.

This section presents the derivations of the relative variance efficiency of estimators (7.15) to (7.14).

Efficiency of the estimates of θ here is simply defined as the quotient of the variances of both estimators.

$$\text{Efficiency} = \frac{\operatorname{var}(\sum_{i=1}^{k} w_i\, r_i)}{\operatorname{var}(k^{-1} \sum_{i=1}^{k} r_i)} \tag{7.16}$$

The weights w_i, $i = 1, 2, \ldots, k$,

$$\text{minimize } \operatorname{var}(\sum_{i=1}^{k} w_i\, r_i)$$

subject to (7.17)

$$\sum_{i=1}^{k} w_i = 1$$

To solve the system equations (7.17), first the following function is formed:

$$f(w_1, \ldots, w_k) = \mathrm{var}(\Sigma\, w_i\, r_i) + \lambda(\Sigma\, w_i - 1)$$

where λ denotes Lagrangian multiplier.

Then the corresponding derivatives with respect to λ and w_i, i = 1, 2, ..., k, are taken, and are set equal to zero to solve for w_i.

The function $f(w_1, \ldots, w_k)$ may be written as

$$f(w_1, \ldots, w_k) = \sum_i w_i^2\, \mathrm{var}(r_i) + \sum_{\substack{i,j \\ i \neq j}} \sum w_i\, w_j\, \mathrm{cov}(r_i, r_j)$$

$$+ \lambda(\sum_i w_i - 1)$$

Its partial derivatives with respect to w_i and λ are then set equal to zero:

$$\frac{\partial f}{\partial w_i} = 2\, w_i\, \mathrm{var}(r_i) + 2 \sum_{j \neq i} w_j\, \mathrm{cov}(r_i, r_j) + \lambda = 0$$

$$\text{for } j \neq i = 1, 2, \ldots, k, \qquad (7.18)$$

$$\frac{\partial f}{\partial \lambda} = \sum_i w_i - 1 = 0$$

The system equations (7.18) can be expressed in the matrix form:

$$\begin{pmatrix} 2\,v_1 & 2\,v_{12} & \cdots & 2\,v_{1k} & 1 \\ 2\,v_{12} & 2\,v_2 & & 2\,v_{2k} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2\,v_{1k} & 2\,v_{2k} & & 2\,v_k & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_k \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix} \tag{7.19}$$

where v_i and v_{ij} denote the variances and covariances of the r_i, respectively. Equation (7.19) may be written as

$$V\underset{\sim}{w} = \underset{\sim}{1}_o$$

If the matrix V is nonsingular, then the weights $\underset{\sim}{w}$ are given by

$$\underset{\sim}{w} = V^{-1}\,\underset{\sim}{1}_o \tag{7.20}$$

For k = 2, the weights w_1 and w_2 are found to be

$$w_1 = \frac{v_{12} - v_2}{2\,v_{12} - v_1 - v_2}$$

and (7.21)

$$w_2 = \frac{v_{12} - v_1}{2\,v_{12} - v_1 - v_2}$$

It can be shown that $\mathrm{var}(w_1\,r_1 + w_2\,r_2)$ is estimated by

$$\frac{v_{12}^2 - v_1 \cdot v_2}{2\, v_{12} - v_1 - v_2}$$

and

$$\mathrm{var}\left(\frac{r_1 + r_2}{2}\right) \text{ by } 1/4\ v_1 + 1/4\ v_2 + v_{12}$$

Hence,

$$\text{Efficiency} = \frac{4(v_{12}^2 - v_1\, v_2)}{(2\, v_{12} - v_1 - v_2)(v_1 + v_2 + v_{12})} \tag{7.22}$$

The following three examples illustrate the computations using (7.22). Consider a property group which is classified into two value categories.

For $k = 1$, $\ell = 2$, and $M = 2$, (7.2) gives

$$\begin{aligned}\mathrm{cov}(r_1, r_2) &= (a\ \pi_a + b\ \pi_b)^{-3} \times \\ &\times \left[\frac{(a\ \pi_a)^2}{n_a} (b\ \pi_b)\ p_a(p_b - p_a) + \right. \\ &\left. + \frac{(b\ \pi_b)^2}{n_b} (a\ \pi_a)\ p_b(p_a - p_b)\right]\end{aligned} \tag{7.23}$$

Equation (7.3) for $k = 1$ and $k = 2$, respectively, gives,

$$\begin{aligned}\mathrm{var}(r_1) &= (a\ \pi_a + b\ \pi_b)^{-2} \left((a\ \pi_a)^2 \frac{p_a}{n_a} + (b\ \pi_b)^2 \frac{p_b}{n_b} - \right. \\ &\left. - 2(a\ \pi_a)^2 \frac{p_a^2}{n_a} - 2(b\ \pi_b)^2 \frac{p_b^2}{n_b}\right)\end{aligned}$$

and (7.24)

$$\mathrm{var}(r_2) = (a\ \pi_a + b\ \pi_b)^{-2} \left((a\ \pi_a)^2 \frac{p_a}{n_a} + (b\ \pi_b)^2 \frac{p_b}{n_b}\right) +$$

$$+ (a\ \pi_a + b\ \pi_b)^{-3} \left(2(a\ \pi_a)^2 (b\ \pi_b) \frac{p_a\ p_b}{n_a} + 2(a\ \pi_a)(b\ \pi_b)^2 \frac{p_a\ p_b}{n_b} -\right.$$

$$\left. - ((a\ \pi_a)^3 + 3(a\ \pi_a)^2 (b\ \pi_b)) \frac{p_a^2}{n_a} - (3(a\ \pi_a)(b\ \pi_b)^2 + (b\ \pi_b)^3) \frac{p_b^2}{n_b}\right)$$

Of special interest here is the case in which the terms of order two are negligible. It follows from (7.23) and (7.24), respectively, that

$$\mathrm{cov}(r_1,\ r_2) = 0$$

$$v_1 = \mathrm{var}(r_1) = (a\ \pi_a + b\ \pi_b)^{-2} \left((a\ \pi_a)^2 \frac{p_a}{n_a} + (b\ \pi_b)^2 \frac{p_b}{n_b}\right)$$

and (7.25)

$$v_2 = \mathrm{var}(r_2) = (a\ \pi_a + b\ \pi_b)^{-2} \left((a\ \pi_a)^2 \frac{p_a}{n_a} + (b\ \pi_b)^2 \frac{p_b}{n_b}\right)$$

Upon the substitution of (7.25) into (7.22), the expression yields

$$\text{Efficiency} = \frac{4\ v_1\ v_2}{(v_1 + v_2)^2}$$

$$= \frac{4(a\ \pi_a + b\ \pi_b)^{-4} (n_a^{-1}(a\ \pi_a)^2\ p_a + n_b^{-1}(b\ \pi_b)^2\ p_b)^2}{(2(a\ \pi_a + b\ \pi_b)^{-2} (n_a^{-1}(a\ \pi_a)^2\ p_a + n_b^{-1}(b\ \pi_b)^2\ p_b)^2}$$

$$= 1$$

Example 2.

Consider mortality data aggregated over several vintage groups which are based on item units. For k = 1 and ℓ = 2, equation (7.7) gives

$$\text{cov}(r_1, r_2) = \sum_{uw}\sum \pi_{e-w,1}\, \pi_{e-w,2}\, \pi_{e-u,2}\, p_{e-w}(p_{e-u} - p_{e-w}). \qquad (7.26)$$

From equation (7.10), it follows that:

$$\text{var}(r_1) = \sum_w \frac{\pi^2_{e-w+1,1}}{e_{e-w+1}} p_{e-w+1}\, q_{e-w+1}$$

$$\text{var}(r_2) = \sum_w \frac{\pi^2_{e-w,2}}{n_{e-w}} p_{e-w}\, q_{e-w} - 2 \sum_{uw}\sum \frac{\pi_{e-u,2}\, \pi^2_{e-w,2}}{n_{e-w}} p_{e-w}\, p_{e-u}).$$

Under the condition that the second order terms are negligible:

$$\text{cov}(r_1, r_2) = 0$$

$$\text{var}(r_1) = \sum_w \frac{\pi^2_{e-w+1,1}}{n_{e-w+1}} p_{e-w+1}$$

and (7.27)

$$\text{var}(r_2) = \sum_w \frac{\pi^2_{e-w,2}}{n_{e-w}} p_{e-w}.$$

Upon the substitution of (7.26) into (7.22), the expression yields

$$\text{Efficiency} = \frac{4\left(\sum_w \frac{\pi^2_{e-w+1,1}}{n_{e-w+1}} p_{e-w+1}\right)\left(\sum_w \frac{\pi^2_{e-w,2}\, p_{e-w}}{n_{e-w}}\right)}{\left(\sum_w \frac{\pi^2_{e-w+1,1}\, p_{e-w+1}}{n_{e-w+1}} + \sum_w \frac{\pi^2_{e-w,2}\, p_{e-w}}{n_{e-w}}\right)^2}$$

Example 3.

Consider mortality data aggregated over several vintage groups which are based on units of dollars. For k = 1 and ℓ = 2, equation (7.11) gives

$$\mathrm{cov}(r_1, r_2) = (\sum_u a_{e-u+1}\, \pi_{e-u+1,1})^{-2} (\sum_v a_{e-v}\, \pi_{e-v,2})^{-2} \times$$

$$\times [\sum_{stw}\sum\sum \frac{a_{e-s+1}\, a_{e-t+1}}{n_{e-w}}\, a^2_{e-w}\, \pi_{e-s+1,1}\, \pi_{e-t,2}\, \pi_{e-w,1} \times$$

$$\times\, \pi_{e-w,2}\, p_{e-w}(p_{e-t} - p_{e-w})].$$

For k = 1 and k = 2 equation (7.12) gives, respectively:

$$\mathrm{var}(r_1) = (\sum_u a_{e-u+1}\, \pi_{e-u+1,1})^{-4} [\sum_{stw}\sum\sum \frac{a_{e-s+1}\, a_{e-t+1}}{n_{e-w+1}} \times$$

$$\times\, a^2_{e-w+1}\, \pi_{e-s+1,1}\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1}\, q_{e-w+1}]$$

and

$$\mathrm{var}(r_2) = (\sum_u a_{e-u}\, \pi_{e-u,2})^{-4}$$

$$\times [\sum_{stw}\sum\sum \frac{a_{e-s}\, a_{e-t}\, a^2_{e-w}}{n_{e-w}}\, \pi_{e-s,2}\, \pi_{e-t,2}\, \pi^2_{e-w,2}\, p_{e-w}\, q_{e-w} -$$

$$- 2 \sum_{svw}\sum\sum \frac{a_{e-s}\, a_{e-v}\, a^2_{e-w}}{n_{e-w}}\, \pi_{e-w,2}\, \pi_{e-v,2}\, \pi^2_{e-w,2}\, p_{e-w}\, p_{e-v}]$$

Of special interest here is the case in which the terms of order two are negligible.

Under this condition:

$$\mathrm{cov}(r_1, r_2) = 0$$

$$\mathrm{var}(r_1) = (\sum_u a_{e-u+1}\, \pi_{e-u+1,1})^{-4} \times$$

$$\times \left[\sum_{stw}\sum\sum \frac{a_{e-s+1}\, a_{e-t+1}}{n_{e-w+1}}\, a^2_{e-w+1}\, \pi_{e-s+1,1}\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1}\right] \qquad (7.28)$$

and

$$\mathrm{var}(r_2) = (\sum_u a_{e-u}\, \pi_{e-u,2})^{-4} \times$$

$$\times \left[\sum_{stw}\sum\sum \frac{a_{e-s}\, a_{e-t}\, a^2_{e-w}}{n_{e-w}}\, \pi_{e-s,2}\, \pi_{e-t,2}\, \pi^2_{e-w,2}\, p_{e-w}\right].$$

Upon the substitution of (7.28) into (7.22), the expression yields

$$\text{Efficiency} = 4(\sum_u a_{e-u+1}\, \pi_{e-u+1,1})^{-4}\, (\sum_u a_{e-u}\, \pi_{e-u,2})^{-4} \times$$

$$\times \left(\sum_{stw}\sum\sum \frac{a_{e-s+1}\, a_{e-t+1}}{n_{e-w+1}}\, a^2_{e-w+1}\, \pi_{e-s+1,1}\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1}\right) \times$$

$$\times \left(\sum_{stw}\sum\sum \frac{a_{e-s}\, a_{e-t}\, a^2_{e-w}}{n_{e-w}}\, \pi_{e-s,2}\, \pi_{e-t,2}\, \pi^2_{e-w,2}\, p_{e-w}\right) \times$$

$$\times \left[(\sum_u a_{e-u+1}\, \pi_{e-u+1,1})^{-4} \left(\sum_{stw}\sum\sum \frac{a_{e-s+1}\, a_{e-t+1}}{n_{e-w+1}}\, a^2_{e-w+1}\, \pi_{e-s+1,1} \times\right.\right.$$

$$\left.\times\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1}\right) +$$

$$+ (\sum_u a_{e-u} \pi_{e-u,2})^{-4} (\sum\sum\sum_{stw} \frac{a_{e-s} a_{e-t} a^2_{e-w}}{n_{e-w}} \pi_{e-s,2} \pi_{e-t,2} \pi^2_{e-w,2} p_{e-w})]^{-2}$$

D. Evaluating Bias

In section B of this chapter, the true average retirement rates over certain age intervals, $\theta = \frac{1}{k} \sum_{i=1}^{k} E(r_i)$, are estimated by OLS and WLS resulting in the estimators $\hat{\theta}_1 = \frac{1}{k} \sum_{i=1}^{k} r_i$ and $\hat{\theta}_2 = \sum_{i=1}^{k} w_i r_i$, respectively. Estimator $\hat{\theta}_1$ will be an unbiased estimator for θ, while $\hat{\theta}_2$ will be biased, with the bias given by

$$\text{Bias} = \sum_{i=1}^{k} (w_i - 1/k) E(r_i). \tag{7.29}$$

For k = 2, with $v_{12} = 0$ to the first order in p, (7.29) may be expressed as

$$\text{Bias} = \frac{v_2(E(r_1) - E(r_2)) + v_1(E(r_2) - E(r_1))}{2(v_1 + v_2)} . \tag{7.30}$$

This section concentrates on evaluating (7.29) and (7.30).

1. Property groups classified by value and life

Before bias can be computed, it is necessary to evaluate $E(r_k)$. To obtain a better understanding consider the property group which is classified into two value-categories, say a and b.

Equation (5.5) gives

$$r_k = \frac{\lambda_k}{(\lambda - \sum_i \lambda_i)} + \frac{\varepsilon_k}{(\lambda - \sum_i \lambda_i)} + \lambda_k \frac{(\sum_i \varepsilon_i)}{(\lambda - \sum_i \lambda_i)^2} . \tag{7.31}$$

Taking the expectation of (7.31), one obtains

$$E(r_k) \doteq \frac{\lambda_k}{(\lambda - \sum_i \lambda_i)} \tag{7.32}$$

Since by definition $E(\varepsilon_i) = 0$ for all i.

In terms of p's, (7.32) can be written as

$$E(r_k) = \frac{a\,\pi_a\,p_{a_k} + b\,\pi_b\,p_{b_k}}{(a\,\pi_a + b\,\pi_b - a\,\pi_a \sum_i p_{a_i} - b\,\pi_b \sum_i p_{b_i})} \,. \tag{7.33}$$

But

$$(a\,\pi_a + b\,\pi_b)^{-1} \left(1 - \frac{a\,\pi_a \sum_i p_{a_i} + b\,\pi_b \sum_i p_{b_i}}{a\,\pi_a + b\,\pi_b}\right)^{-1} =$$

$$(a\,\pi_a + b\,\pi_b)^{-1} \left(1 + \frac{a\,\pi_a \sum_i p_{a_i} + b\,\pi_b \sum_i p_{b_i}}{a\,\pi_a + b\,\pi_b} + \ldots\right) \tag{7.34}$$

Upon the substitution of (7.34) into (7.33), the expression yields

$$E(r_k) = \frac{a\,\pi_a\,p_{a_k} + b\,\pi_b\,p_{b_k}}{a\,\pi_a + b\,\pi_b} \left(1 + \frac{a\,\pi_a \sum_i p_{a_i} + b\,\pi_b \sum_i p_{b_i}}{a\,\pi_a + b\,\pi_b} + \ldots\right)$$

$$E(r_k) = \frac{a\,\pi_a\,p_{a_k} + b\,\pi_b\,p_{b_k}}{a\,\pi_a + b\,\pi_b} +$$

$$+ \frac{(a\,\pi_a)^2\,p_{a_k} \sum_i p_{a_i} + (b\,\pi_b)^2\,p_{b_k} \sum_i p_{b_i}}{(a\,\pi_a + b\,\pi_b)^2} +$$

$$+ \frac{(a\,\pi_a)(b\,\pi_b)(p_{a_k} \sum_i p_{b_i} + p_{b_k} \sum_i p_{a_i})}{(a\,\pi_a + b\,\pi_b)^2} + \ldots \tag{7.35}$$

In the case of a geometric distribution,

$$p_{i_k} = p_i(1 - p_i)^k$$

and when p's are small,

$$p_{i_k} \doteq p_i(1 - k\, p_i) \tag{7.36}$$

for i = a, b.

Upon the substitution of (7.36) into (7.35) and when the third order terms are negligible, it can be shown that:

$$E(r_k) = (a\,\pi_a + b\,\pi_b)^{-1}(a\,\pi_a\,p_a + b\,\pi_b\,p_b) - (a\,\pi_a + b\,\pi_b)^{-2} \times$$

$$\times\ ((a\,\pi_a)^2\,p_a^{\ 2} + (b\,\pi_b)^2\,p_b^{\ 2} + k(a\,\pi_a)(b\,\pi_b)(p_a^{\ 2} + p_b^{\ 2}) -$$

$$- 2(k - 1)(a\,\pi_a)(b\,\pi_b)\,p_a\,p_b). \tag{7.37}$$

Under the conditions that the terms of order two are negligible:

$$E(r_k) = (a\,\pi_a + b\,\pi_b)^{-1}(a\,\pi_a\,p_a + b\,\pi_b\,p_b)\,.$$

Also from (7.25) it follows that:

$$\mathrm{cov}(r_1, r_2) = 0$$

$$\mathrm{var}(r_1) = (a\,\pi_a + b\,\pi_b)^{-2}\,((a\,\pi_a)^2\,\frac{p_a}{n_a} + (b\,\pi_b)^2\,\frac{p_b}{n_b})$$

$$\mathrm{var}(r_2) = (a\,\pi_a + b\,\pi_b)^{-2}\,((a\,\pi_a)^2\,\frac{p_a}{n_a} + (b\,\pi_b)^2\,\frac{p_b}{n_b}) \qquad (7.38)$$

Upon the substitution of (7.38) into (7.21), the expression yields

$$w_1 = \frac{(a\,\pi_a + b\,\pi_b)^{-2}\,((a\,\pi_a)^2\,\frac{p_a}{n_a} + (b\,\pi_b)^2\,\frac{p_b}{n_b})}{2(a\,\pi_a + b\,\pi_b)^{-2}\,((a\,\pi_a)^2\,\frac{p_a}{n_a} + (b\,\pi_b)^2\,\frac{p_b}{n_b})} = \frac{1}{2}$$

and

$$w_2 = \frac{(a\,\pi_a + b\,\pi_b)^{-2}\,((a\,\pi_a)^2\,\frac{p_a}{n_a} + (b\,\pi_b)^2\,\frac{p_b}{n_b})}{2(a\,\pi_a + b\,\pi_b)^{-2}\,((a\,\pi_a)^2\,\frac{p_a}{n_a} + (b\,\pi_b)^2\,\frac{p_b}{n_b})} = \frac{1}{2}$$

Hence,

$$\text{Bias} = \sum_{i=1}^{2} (w_i - 1/2)\,E(r_i) = 0 .$$

Observe that when the contributions of the terms of order two are significant, the bias is generally not zero.

2. Property groups classified by vintage and life

Bias is evaluated as follows for the case of property group classified into multivintages and life. When data are based on unit counts, equation (6.40) gives

$$r_k = \frac{\lambda_{e-L-k+2,k}}{(1 - \sum_i \lambda_{e-L-k+2,i})} + \frac{\varepsilon_{e-L-k+2,k}}{(1 - \sum_i \lambda_{e-L-k+2,i})} +$$

$$+ \frac{\lambda_{e-L-k+2,k}(\sum_i \varepsilon_{e-L-k+2,i})}{(1 - \sum_i \lambda_{e-L-k+2,i})^2} . \qquad (7.39)$$

The expectation of (7.39) yields

$$E(r_k) = \frac{\lambda_{e-L-k+2,k}}{(1 - \sum_i \lambda_{e-L-k+2,i})} \qquad (7.40)$$

since by definition, $E(\varepsilon) = 0$.

$$E(r_k) = \frac{\sum_w \pi_{e-w-k+2,k}\, p_{e-w-k+2,k}}{(1 - \sum\sum_{iw} \pi_{e-w-k+2,k}\, p_{e-w-k+2,i})} \qquad (7.41)$$

But

$$(1 - \sum\sum_{iw} \pi_{e-w-k+2,k}\, p_{e-w-k+2,i})^{-1} =$$

$$= 1 + \sum\sum_{iw} \pi_{e-w-k+2,k}\, p_{e-w-k+2,i} + \cdots \qquad (7.42)$$

The substitution of equation (7.42) into equation (7.41) yields

$$E(r_k) = \sum_w \pi_{e-w-k+2,k}\, p_{e-w-k+2,k} +$$

$$+ \sum\sum\sum_{iuw} \pi_{e-u-k+2,k}\, \pi_{e-w-k+2,k}\, p_{e-u-k+2,k}\, p_{e-w-k+2,i} + \cdots \qquad (7.43)$$

The retirement experience from the vintage groups included in the study is assumed to follow geometric distributions, i.e.,

$$p_{e-w-i+2,i} = p_{e-w-i+2}(1 - i\, p_{e-w-i+2})$$

for all i and w.

Under the above condition and when the terms of order three are negligible, equation (7.43) is simplified to:

$$\begin{aligned} E(r_k) &= \sum_w \pi_{e-w-k+2,k}\, p_{e-w-k+2} - \\ &\quad - k \sum_w \pi_{e-w-k+2,k}\, p^2_{e-w-k+2} + \\ &\quad + (k-1) \sum_{uw}\sum \pi_{e-u-k+2,k}\, \pi_{e-w-k+2,k}\, p_{e-u-k+2}\, p_{e-w-k+2}. \end{aligned} \tag{7.44}$$

Of special interest here is the case in which the terms of order two are also negligible due to the small p's. Equation (7.44) then gives

$$E(r_k) = \sum_w \pi_{e-w-k+2,k}\, p_{e-w-k+2}. \tag{7.45}$$

Upon the substitution of (7.27) and (7.45) into (7.30), the bias can be estimated by

$$\text{Bias} = \frac{1}{2}\left(\sum_w \frac{\pi^2_{e-w+1,1}\, p_{e-w+1}}{n_{e-w+1}} + \sum_w \frac{\pi^2_{e-w,2}\, p_{e-w}}{n_{e-w}}\right)^{-1} \times$$

$$\times \left[\sum_w \frac{\pi^2_{e-w,2}\, p_{e-w}}{n_{e-w}} \left(\sum_w (\pi_{e-w+1,1}\, p_{e-w+1} - \pi_{e-w,2}\, p_{e-w}\right)\right) +$$

$$+ \sum_w \frac{\pi^2_{e-w+1,1}\, p_{e-w+1}}{n_{e-w+1}} \left(\sum_w (\pi_{e-w,2}\, p_{e-w} - \pi_{e-w+1,1}\, p_{e-w+1})\right)\Big] \qquad (7.46)$$

The bias of estimator $\hat{\theta}_2$ is next computed for the case of data measured based on dollars. Equation (6.70) gives

$$r_k = \frac{\lambda_{e-w-k+2,k}}{(\lambda_{*k} - \sum_i \lambda_{e-w-k+2,i})} + \frac{\varepsilon_{e-w-k+2,k}}{(\lambda_{*k} - \sum_w \lambda_{e-w-k+2,i})}$$

$$+ \lambda_{e-w-k+2,k} \frac{(\sum_i \varepsilon_{e-w-k+2,i})}{(\lambda_{*k} - \sum_i \lambda_{e-w-k+2,i})^2}$$

As before, under geometric conditional distribution, it can be shown that:

$$E(r_k) = (\sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k})^{-1} \times$$

$$\times (\sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k}\, p_{e-w-k+2}) +$$

$$+ (k-1)(\sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k})^{-2} \sum_{vw}\sum a_{e-v-k+2}\, a_{e-w-k+2} \times$$

$$\times \pi_{e-v-k+2,k}\, \pi_{e-w-k+2,k}\, p_{e-v-k+2}\, p_{e-w-k+2}\,. \qquad (7.47)$$

Furthermore, when the second order terms are ignored, $E(r_k)$ is given by

$$E(r_k) \doteq (\sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k})^{-1}$$

$$\times (\sum_w a_{e-w-k+2}\, \pi_{e-w-k+2,k}\, p_{e-w-k+2}) \qquad (7.48)$$

for $k = 1, 2, \ldots$

Under the condition that the second order terms are negligible, equation (7.28) gives

$$\text{cov}(r_1, r_2) = 0.$$

$$\text{var}(r_1) = (\sum_w a_{e-w+1}\, \pi_{e-w+1,1})^{-4}$$

$$\times [\sum_{stw}\sum\sum n^{-1}_{e-w+1}\, a_{e-s+1}\, a_{e-t+1}\, a^2_{e-w+1}\, \pi_{e-s+1,1}\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1}]$$

and

$$\text{var}(r_2) = (\sum_w a_{e-w+1}\, \pi_{e-w+1,1})^{-4}$$

$$\times [\sum_{stw}\sum\sum n^{-1}_{e-w+1}\, a_{e-s+1}\, a_{e-t+1}\, a^2_{e-w+1}\, \pi_{e-s+1,1}\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1}] \qquad (7.49)$$

Upon the substitution of (7.48) and (7.49) into (7.30), the bias of the estimator $\hat{\theta}_2$ is given by

$$\text{Bias} = \frac{1}{2}[(\sum_u a_{e-u+1}\, \pi_{e-u+1,1})^{-4}\ (\sum_{stw}\sum\sum \frac{a_{e-s+1}\, a_{e-t+1}\, a^2_{e-w+1}}{n_{e-w+1}} \times$$

$$\times \pi_{e-s+1,1}\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1} +$$

$$+ (\sum_u a_{e-u}\, \pi_{e-u,2})^{-4} (\sum_{stw}\sum\sum \frac{a_{e-s}\, a_{e-t}\, a^2_{e-t}}{n_{e-w}}\, \pi_{e-s,2}\, \pi_{e-t,2}\, \pi^2_{e-w,2}\, p_{e-w})]^{-1} \times$$

$$\times [(\sum_u a_{e-u}\, \pi_{e-u,2})^{-4} (\sum_{stw}\sum\sum \frac{a_{e-s}\, a_{e-t}\, a^2_{e-w}}{n_{e-w}}\, \pi_{e-s,2}\, \pi_{e-t,2}\, \pi^2_{e-w,2}\, p_{e-w}) \times$$

$$\times ((\sum_u a_{e-w+1}\, \pi_{e-w+1,1})^{-1} (\sum_w a_{e-w+1}\, \pi_{e-w+1,1}\, p_{e-w+1}) -$$

$$- (\sum_w a_{e-w}\, \pi_{e-w,2})^{-1} (\sum_w a_{e-w}\, \pi_{e-w,2}\, p_{e-w})) +$$

$$+ (\sum_u a_{e-u+1}\, \pi_{e-u+1,1})^{-4} (\sum_{stw}\sum\sum \frac{a_{e-s+1}\, a_{e-t+1}}{n_{e-w+1}} \times$$

$$\times a^2_{e-w+1}\, \pi_{e-s+1,1}\, \pi_{e-t+1,1}\, \pi^2_{e-w+1,1}\, p_{e-w+1}) \times$$

$$\times ((\sum_w a_{e-w}\, \pi_{e-w,2})^{-1} (\sum_w a_{e-w}\, \pi_{e-w,2}\, p_{e-w}) -$$

$$- (\sum_w a_{e-w+1}\, \pi_{e-w+1,1})^{-1} (\sum_w a_{e-w+1}\, \pi_{e-w+1,1}\, p_{e-w+1}))]. \qquad (7.50)$$

SUMMARY AND CONCLUSIONS

The univariate distributions used in previous research do not take into consideration the random variability of life (age). Value is not independent of ages; hence, the univariate distributions can not fully describe the relationship between value and life. The above facts lead to the development of bivariate distributions of value and life.

In this study, the joint continuous and discrete distributions of value and life were modeled. For the case of joint continuous distribution, bivariate lognormal and gamma distributions were applied to represent F(t), the proportion of dollars surviving up to age t. These distributions are well-known, but their application to life analysis appears to be new.

When the joint distribution is bivariate lognormal, F(t) can be represented by

$$F(t) = 1 - \Phi\left(\frac{\ell n\ t - \mu_2 - \sigma_{12}}{\sigma_2}\right) .$$

Under the bivariate lognormal, the estimate of F(t) can be found by simply replacing unknown quantities by the sample quantities;

$$F(\hat{t}) = 1 - \Phi\left(\frac{\ell n\ t - \hat{\mu}_2 - \hat{\sigma}_{12}}{\hat{\sigma}_2}\right) .$$

So, when the mean, variance and covariance of the samples are known, the proportion of dollars surviving up to any age can be computed. Under the bivariate gamma distribution, F(t) is given by

$$F(t) = \frac{a(1 - \Gamma_{a+c+1}(t)) + b(1 - \Gamma_{a+c}(t))}{a + b}$$

For the case of joint discrete distribution, the asymptotic covariance and variance structures of the retirement ratios of the industrial properties which are respectively subject to the following conditions, were determined.

1. The retirement experience from value or vintage groups have different mortality characteristics, i.e., multinomial life distributions.

2. The retirement experience from value or vintage groups have the same mortality law.

3. The retirement experience from value or vintage groups are assumed to follow geometric distributions, for both the situation 1., and the situation 2.

Under 1., it was found that the asymptotic covariances between the retirement ratios for two different age intervals are generally not zero.

Under 2., it can be shown that the asymptotic covariances between the retirement ratios for two nonoverlapping age intervals are zero. Therefore, the retirement ratios are uncorrelated. Chiang (1960a) established zero correlation of retirement ratios for small samples, using a different method. Since a multinomial distribution tends to normality for large sample sizes, then the retirement ratios are found to be asymptotically independent.

When the size of all vintage groups are taken to be equal, the asymptotic variances of retirement ratios can be written explicitly as a function of the inverse of the width of the experience band used in the study and the size of vintage. In general, it is true that the asymptotic variances of retirement ratios are inversely related to the band width and vintage size.

This relationship may explain the basic idea of choosing the width of experience band to be between 3 and 30 years, which is recommended by Marston et al. (1979). The smaller variances serve to reduce errors in estimating the survivor function developed on the basis of those retirement ratios.

When item values are incorporated in the analysis, the magnitude of the asymptotic variances of retirement ratios is reduced. Hence, the primary effect of using dollars as measures of the amount of property is to reduce the magnitude of the asymptotic variances.

The condition that all value or vintage groups die according to the same mortality law may not be a realistic assumption. Industrial properties and the nature of their retirements are very complex. Many factors influence the rates of retirement in different ways and may have dissimilar effects on the various property groups. However, if the condition is assumed and the number of units from each value or vintage group is sufficiently large, the retirement ratios are nearly independent. Further, because of the characteristics of approximate independence of the retirement ratios, it is sufficient for fitting a general linear model to retirement ratios to use weighted least squares with only diagonal terms. Various weighting procedures can be developed for the above purposes.

Mortality characteristic for long-lived property can be represented by a geometric distribution having a small parameter p. Under this distributional assumption the asymptotic variances and covariances are much simplified when the contributions of the third order terms of p's are insignificant. Of special interest here is the case in which the second order p-terms are also negligible; for a single vintage group having two value categories, and the case of approximately constant early retirement rates, it is found that both the unweighted estimator $\hat{\theta}_1 = \frac{1}{k} \sum_{i=1}^{k} r_i$ and the weighted estimator $\hat{\theta}_2 = \sum_{i=1}^{k} w_i r_i$, for $k = 2$, have the same variance. For common industrial mortality data aggregated over several vintage groups, both estimators do not have the same approximate variance. The approximate biases of the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ were also computed.

It would be interesting for future investigations to apply the technique of asymptotic expansion of distributions to develop higher order error terms for the linear-normal approximations used in the development of the asymptotic distributions developed in this thesis.

XI. LITERATURE CITED

Anderson, T. W. 1958. An introduction to multivariate statistical analysis. John Wiley & Sons, Inc., New York, N.Y.

Bissinger, B. H. 1981. A long life analysis. The Statistician 30:15-21.

Buehler, Robert J. 1960. An application of regression to frequency graduation. Biometrics 16:659-670.

Chiang, C. L. 1960a. A stochastic study of the life table and its applications: I. Probability distributions of the biometric functions. Biometrics 16:619-635.

Chiang, C. L. 1960b. A stochastic study of the life table and its applications: II. Sample variance of the observed expectation of life and other biometric functions. Human Biology 32:221-238.

Cohen, A. C. 1951. Estimating parameters of logarithmic-normal distributions by maximum likelihood. Journal of the American Statistical Association 46:207-212.

Couch, Frank Van Buskirk, Jr. 1957. Classification of type O retirement characteristics of industrial property. Unpublished M.S. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

Cowles, Harold Andrews, Jr. 1957. Prediction of mortality characteristics of industrial property groups. Unpublished Ph.D. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

Cox, D. R. Regression models and life tables. Journal of the Royal Statistical Society Series B, 34, No. 2:187-220. 1972.

Edison Electric Institute. 1952. Methods of estimating utility plant life. Edison Electric Institute, New York, N.Y.

Feller, William. 1950. An introduction to probability theory and its applications. John Wiley and Sons, Inc., New York, N.Y.

Fleiss, Joseph L. 1982. Simplification of the classic large-sample standard error of a function of multinomial proportions. The American Statistician 34, No. 4:377.

Ghirtis, G. C. 1967. Some problems of statistical inference relating to the double-gamma distribution. Trabajos de Estadistica No. 18:67-87.

Gross, Allen and Clark, William. 1975. Survival distributions-reliability applications in the biomedical sciences. John Wiley and Sons, Inc., New York, N.Y.

Gumbel, E. J. 1960. Bivariate exponential distributions. Journal of the American Statistical Association 55:699-707.

Henderson, Allen James. 1965. The Weibull distribution and industrial property mortality experience. Unpublished M.S. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

Henderson, Allen James. 1968. Actuarial methods for estimating property parameters of industrial property. Unpublished Ph.D. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

Jeynes, P. H. 1947. Indicated renewals. Unpublished multilithed paper presented at National Conference of Electrical and Gas Utility Accountants American Gas Association-Edison Electrical Institute, Buffalo, New York, April 8, 1947. Public Service Electric and Gas Company, Newark, New Jersey.

Johnson, N., and Kotz, S. 1969. Discrete Distributions (Vol. 1). John Wiley & Sons, New York, N.Y.

Johnson, Elandt, Regina C. and Johson, Norman L. 1980. Survival models and data analysis. John Wiley & Sons, New York, N.Y.

Kalbfleisch, John D. and Prentice, Ross L. 1980. The statistical analysis of failure time data. John Wiley & Sons, New York, N.Y.

Kaplan, E. L. and Meier, Paul. 1958. Nonparametric estimation from incomplete observations. Journal of the American Statistical Association 53:457-481.

Kimball, Bradford F. 1947. A system of life tables for phsyical property based on the truncated normal distribution. Econometrica 15:342-360.

Krane, Scott A. 1963. Analysis of survival data by regression techniques. Technometrics 5, No. 2:161-174.

Kurtz, Edwin B. 1930. Life expectancy of physical property. The Ronald Press Company, New York, N.Y.

Lamp, George Emmett, Jr. 1968. Dispersion effects in industrial property life analysis. Unpublished Ph.D. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

Lawless, Jerald F. 1982. Statistical models and methods for lifetime data. John Wiley & Sons, New York, N.Y.

Lindgren, B. W. 1968. Statistical Theory, 3rd ed. Macmillan, New York, N.Y.

Marston, Anson, Winfrey, Robley, and Hempstead, Jean C. 1979. Engineering valuation and depreciation. Iowa State University Press, Ames, Iowa.

Miller, Morton D. 1946. Elements of graduation. Actuarial Monograph no. 1, The Actuarial Society of America and American Institute of Actuaries, Chicago, Illinois.

Mood, Alexander M., Graybill, Franklin A., and Boes, Duane C. 1974. Introduction to the Theory of Statistics. McGraw Hill, New York, N.Y.

Moran, P. A. P. 1969. Statistical inference with bivariate gamma distributions. Biometrika 56:627-634.

Nelson, Wayne. 1972. Theory and applications of hazard plotting for censored failure data. Technometrics 14:945-966.

Nelson, Wayne. 1982. Applied life data analysis. John Wiley & Sons, New York, N.Y.

Nichols, Richard Lee. 1961. The moment ratio method of analyzing industrial property experience. Unpublished M.S. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

Ostle, Bernard. 1964. Statistics in research. Iowa State University Press, Ames, Iowa.

Plackett, R. L. 1965. A class of bivariate distributions. Journal of the American Statistical Association 60:510-522.

White, Ronald Eugene. 1968. The multi-variate normal distribution and the simulated plant record method of life analysis. Unpublished M.S. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

White, R. E. 1977. A comparative analysis of various estimates of the hazard rate associated with the service life of industrial property. Unpublished Ph.D. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa.

Winfrey, Robley. 1942. Depreciation of group properties. Iowa State College Engineering Experiment Station Bulletin 155.

Winfrey, Robley. 1967. Statistical analyses of industrial property retirements: revised April, 1967 by H. A. Cowles, Professor, Department of Industrial Engineering. Iowa State University Engineering Research Bulletin 125, revised edition.

Winfrey, Robley and Kurtz, Edwin B. 1931. Life characteristics of physical property. Iowa State College Engineering Experiment Station Bulletin 103.

X. ACKNOWLEDGMENTS

I would like to express my sincere thanks to Professor Harold A. Cowles for his encouragement and guidance throughout my graduate career. I would like to express a sincere debt of appreciation to Professor Herbert T. David from Statistical Laboratory for his generosity in time and assistance in developing the statistical models. Without his support, the research which culminates in this dissertation, would not have been possible.

A debt of gratitude is also due to the other members of my graduate committee, Professors Gerald W. Smith, Lehman B. Fletcher and Howard D. Meeks.

The financial assistance and support received from the Indonesian Government was also instrumental in making this study possible.

Last, but certainly not least, I wish to thank my wife, Winarsih and my son, Wahyudi, whose patience and encouragement were a major factor in this study.